Two-dimensional shapes such as quadrilaterals and circles can be used to describe and model the world around us. In this unit, you will learn about the properties of quadrilaterals and circles and how these two-dimensional figures can be transformed.
“Geocaching” Sends Folks on a Scavenger Hunt

“N42 DEGREES 02.054 W88 DEGREES 12.329 – Forget the poison ivy and needle-sharp brambles. Dave April is a man on a mission. Clutching a palm-size Global Positioning System (GPS) receiver in one hand and a computer printout with latitude and longitude coordinates in the other, the 37-year-old software developer trudges doggedly through a suburban Chicago forest preserve, intent on finding a geek’s version of buried treasure.” Geocaching is one of the many new ways that people are spending their leisure time. In this project, you will use quadrilaterals, circles, and geometric transformations to give clues for a treasure hunt.

Log on to www.geometryonline.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 3.
Several different geometric shapes are examples of quadrilaterals. These shapes each have individual characteristics. A rectangle is a type of quadrilateral. Tennis courts are rectangles, and the properties of the rectangular court are used in the game. You will learn more about tennis courts in Lesson 8-4.
Prerequisite Skills  To be successful in this chapter, you’ll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 8.

For Lesson 8-1  Exterior Angles of Triangles
Find x for each figure.  (For review, see Lesson 4-2.)

1. 2. 3.

For Lessons 8-4 and 8-5  Perpendicular Lines
Find the slopes of $\overline{RS}$ and $\overline{TS}$ for the given points, $R$, $T$, and $S$. Determine whether $\overline{RS}$ and $\overline{TS}$ are perpendicular or not perpendicular.  (For review, see Lesson 3-6.)

4. $R(4, 3), S(-1, 10), T(13, 20)$  5. $R(-9, 6), S(3, 8), T(1, 20)$
6. $R(-6, -1), S(5, 3), T(2, 5)$  7. $R(-6, 4), S(-3, 8), T(5, 2)$

For Lesson 8-7  Slope
Write an expression for the slope of a segment given the coordinates of the endpoints.  (For review, see Lesson 3-3.)

8. $\left(\frac{c}{2}, \frac{d}{2}\right), \left(-c, d\right)$  9. $(0, a), (b, 0)$
10. $(-a, c), (-c, a)$

Quadrilaterals  Make this Foldable to help you organize information about quadrilaterals. Begin with a sheet of notebook paper.

Step 1  Fold
Fold lengthwise to the left margin.

Step 2  Cut
Cut 4 tabs.

Step 3  Label
Label the tabs using the lesson concepts.

Reading and Writing  As you read and study the chapter, use your Foldable to take notes, define terms, and record concepts about quadrilaterals.
8-1 Angles of Polygons

What You'll Learn

- Find the sum of the measures of the interior angles of a polygon.
- Find the sum of the measures of the exterior angles of a polygon.

How does a scallop shell illustrate the angles of polygons?

This scallop shell resembles a 12-sided polygon with diagonals drawn from one of the vertices. A diagonal of a polygon is a segment that connects any two nonconsecutive vertices. For example, \(AB\) is one of the diagonals of this polygon.

SUM OF MEASURES OF INTERIOR ANGLES

Polygons with more than three sides have diagonals. The polygons below show all of the possible diagonals drawn from one vertex.

In each case, the polygon is separated into triangles. Each angle of the polygon is made up of one or more angles of triangles. The sum of the measures of the angles of each polygon can be found by adding the measures of the angles of the triangles. Since the sum of the measures of the angles in a triangle is 180, we can easily find this sum. Make a table to find the sum of the angle measures for several convex polygons.

<table>
<thead>
<tr>
<th>Convex Polygon</th>
<th>Number of Sides</th>
<th>Number of Triangle</th>
<th>Sum of Angle Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>3</td>
<td>1</td>
<td>((1 \cdot 180)) or 180</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>4</td>
<td>2</td>
<td>((2 \cdot 180)) or 360</td>
</tr>
<tr>
<td>pentagon</td>
<td>5</td>
<td>3</td>
<td>((3 \cdot 180)) or 540</td>
</tr>
<tr>
<td>hexagon</td>
<td>6</td>
<td>4</td>
<td>((4 \cdot 180)) or 720</td>
</tr>
<tr>
<td>heptagon</td>
<td>7</td>
<td>5</td>
<td>((5 \cdot 180)) or 900</td>
</tr>
<tr>
<td>octagon</td>
<td>8</td>
<td>6</td>
<td>((6 \cdot 180)) or 1080</td>
</tr>
</tbody>
</table>

Look for a pattern in the sum of the angle measures. In each case, the sum of the angle measures is 2 less than the number of sides in the polygon times 180. So in an \(n\)-gon, the sum of the angle measures will be \((n - 2)180\) or \(180(n - 2)\).

Theorem 8.1

Interior Angle Sum Theorem If a convex polygon has \(n\) sides and \(S\) is the sum of the measures of its interior angles, then \(S = 180(n - 2)\).

Example:

\[
\begin{align*}
S &= 180(n - 2) \\
&= 180(5 - 2) \text{ or } 540
\end{align*}
\]
Lesson 8-1  Angles of Polygons

**Example 1  Interior Angles of Regular Polygons**

**CHEMISTRY** The benzene molecule, C₆H₆, consists of six carbon atoms in a regular hexagonal pattern with a hydrogen atom attached to each carbon atom. Find the sum of the measures of the interior angles of the hexagon.

Since the molecule is a convex polygon, we can use the Interior Angle Sum Theorem.

\[ S = 180(n - 2) \quad \text{Interior Angle Sum Theorem} \]

\[ = 180(6 - 2) \quad n = 6 \]

\[ = 180(4) \quad \text{or} \quad 720 \quad \text{Simplify.} \]

The sum of the measures of the interior angles is 720.

The Interior Angle Sum Theorem can also be used to find the number of sides in a regular polygon if you are given the measure of one interior angle.

**Example 2  Sides of a Polygon**

The measure of an interior angle of a regular polygon is 108. Find the number of sides in the polygon.

Use the Interior Angle Sum Theorem to write an equation to solve for \( n \), the number of sides.

\[ S = 180(n - 2) \quad \text{Interior Angle Sum Theorem} \]

\[ (108)n = 180(n - 2) \quad S = 108n \]

\[ 108n = 180n - 360 \quad \text{Distributive Property} \]

\[ 0 = 72n - 360 \quad \text{Subtract 108n from each side.} \]

\[ 360 = 72n \quad \text{Add 360 to each side.} \]

\[ 5 = n \quad \text{Divide each side by 72.} \]

The polygon has 5 sides.

In Example 2, the Interior Angle Sum Theorem was applied to a regular polygon. In Example 3, we will apply this theorem to a quadrilateral that is not a regular polygon.

**Example 3  Interior Angles**

**ALGEBRA** Find the measure of each interior angle.

Since \( n = 4 \), the sum of the measures of the interior angles is 180(4 - 2) or 360. Write an equation to express the sum of the measures of the interior angles of the polygon.

\[ 360 = m \angle A + m \angle B + m \angle C + m \angle D \quad \text{Sum of measures of angles} \]

\[ 360 = x + 2x + 2x + x \quad \text{Substitution} \]

\[ 360 = 6x \quad \text{Combine like terms.} \]

\[ 60 = x \quad \text{Divide each side by 6.} \]

Use the value of \( x \) to find the measure of each angle.

\[ m \angle A = 60, \ m \angle B = 2 \cdot 60 \text{ or } 120, \ m \angle C = 2 \cdot 60 \text{ or } 120, \text{ and } m \angle D = 60. \]
SUM OF MEASURES OF EXTERIOR ANGLES

The Interior Angle Sum Theorem relates the interior angles of a convex polygon to the number of sides. Is there a relationship among the exterior angles of a convex polygon?

### Geometry Activity

**Sum of the Exterior Angles of a Polygon**

**Collect Data**
- Draw a triangle, a convex quadrilateral, a convex pentagon, a convex hexagon, and a convex heptagon.
- Extend the sides of each polygon to form exactly one exterior angle at each vertex.
- Use a protractor to measure each exterior angle of each polygon and record it on your drawing.

**Analyze the Data**

1. Copy and complete the table.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>triangle</th>
<th>quadrilateral</th>
<th>pentagon</th>
<th>hexagon</th>
<th>heptagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of exterior angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sum of measure of exterior angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. What conjecture can you make?

The Geometry Activity suggests Theorem 8.2.

### Theorem 8.2

**Exterior Angle Sum Theorem** If a polygon is convex, then the sum of the measures of the exterior angles, one at each vertex, is 360.

Example:

\[
m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 = 360
\]

You will prove Theorem 8.2 in Exercise 42.

### Example 4 Exterior Angles

Find the measures of an exterior angle and an interior angle of convex regular octagon \(ABCD\EF\GH\).

At each vertex, extend a side to form one exterior angle. The sum of the measures of the exterior angles is 360. A convex regular octagon has 8 congruent exterior angles.

\[
8n = 360 \quad n = \text{measure of each exterior angle}
\]

\[
n = 45 \quad \text{Divide each side by 8.}
\]

The measure of each exterior angle is 45. Since each exterior angle and its corresponding interior angle form a linear pair, the measure of the interior angle is \(180 - 45\) or 135.
**Concept Check**

1. **Explain** why the Interior Angle Sum Theorem and the Exterior Angle Sum Theorem only apply to convex polygons.

2. **Determine** whether the Interior Angle Sum Theorem and the Exterior Angle Sum Theorem apply to polygons that are not regular. Explain.

3. **OPEN ENDED** Draw a regular convex polygon and a convex polygon that is not regular with the same number of sides. Find the sum of the interior angles for each.

**Guided Practice**

Find the sum of the measures of the interior angles of each convex polygon.

4. pentagon

5. dodecagon

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon.

6. 60

7. 90

**ALGEBRA**

Find the measure of each interior angle.

8. 

9. 

Find the measures of an exterior angle and an interior angle given the number of sides of each regular polygon.

10. 6

11. 18

**Application**

12. **AQUARIUMS** The regular polygon at the right is the base of a fish tank. Find the sum of the measures of the interior angles of the pentagon.

Find the sum of the measures of the interior angles of each convex polygon.

13. 32-gon

14. 18-gon

15. 19-gon

16. 27-gon

17. 4y-gon

18. 2x-gon

19. **GARDENING** Carlotta is designing a garden for her backyard. She wants a flower bed shaped like a regular octagon. Find the sum of the measures of the interior angles of the octagon.

20. **GAZEBOS** A company is building regular hexagonal gazebos. Find the sum of the measures of the interior angles of the hexagon.

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon.

21. 140

22. 170

23. 160

24. 165

25. \(157\frac{1}{2}\)

26. \(176\frac{2}{5}\)
ALGEBRA  Find the measure of each interior angle using the given information.

27.  
\[ \begin{align*} \angle M &= 10x \\ \angle P &= 4x \end{align*} \]

29. parallelogram $MNPQ$ with $\angle M = 10x$ and $\angle N = 20x$

30. isosceles trapezoid $TWYZ$ with $\angle Z \equiv \angle Y$, $\angle Z = 30x$, $\angle T \equiv \angle W$, and $\angle T = 20x$

31. decagon in which the measures of the interior angles are $x + 5$, $x + 10$, $x + 20$, $x + 30$, $x + 35$, $x + 40$, $x + 60$, $x + 70$, $x + 80$, and $x + 90$

32. polygon $ABCDE$ with $\angle A = 6x$, $\angle B = 4x + 13$, $\angle C = x + 9$, $\angle D = 2x - 8$, and $\angle E = 4x - 1$

33. quadrilateral in which the measure of each consecutive angle is a consecutive multiple of $x$

34. quadrilateral in which the measure of each consecutive angle increases by 10°

Find the measures of each exterior angle and each interior angle for each regular polygon.

35. decagon

36. hexagon

37. nonagon

38. octagon

Find the measures of an interior angle and an exterior angle given the number of sides of each regular polygon. Round to the nearest tenth if necessary.

39. 11

40. 7

41. 12

42. PROOF  Use algebra to prove the Exterior Angle Sum Theorem.

43. ARCHITECTURE  The Pentagon building in Washington, D.C., was designed to resemble a regular pentagon. Find the measure of an interior angle and an exterior angle of the courtyard.

44. ARCHITECTURE  Compare the dome to the architectural elements on each side of the dome. Are the interior and exterior angles the same? Find the measures of the interior and exterior angles.

45. CRITICAL THINKING  Two formulas can be used to find the measure of an interior angle of a regular polygon: $s = \frac{180(n - 2)}{n}$ and $s = 180 - \frac{360}{n}$. Show that these are equivalent.
46. **Writing in Math** Answer the question that was posed at the beginning of the lesson.

How does a scallop shell illustrate the angles of polygons?

Include the following in your answer:

- explain how triangles are related to the Interior Angle Sum Theorem, and
- describe how to find the measure of an exterior angle of a polygon.

47. A regular pentagon and a square share a mutual vertex $X$. The sides $XY$ and $XZ$ are sides of a third regular polygon with a vertex at $X$. How many sides does this polygon have?

- A 19
- B 20
- C 28
- D 32

48. **Grid In** If $6x + 3y = 48$ and $\frac{9y}{2x} = 9$, then $x =$ ?

**Maintain Your Skills**

**Mixed Review** In $\triangle ABC$, given the lengths of the sides, find the measure of the given angle to the nearest tenth. **(Lesson 7-7)**

- 49. $a = 6, b = 9, c = 11; m\angle C$
- 50. $a = 15.5, b = 23.6, c = 25.1; m\angle B$
- 51. $a = 47, b = 53, c = 56; m\angle A$
- 52. $a = 12, b = 14, c = 16; m\angle C$

Solve each $\triangle FGH$ described below. Round angle measures to the nearest degree and side measures to the nearest tenth. **(Lesson 7-6)**

- 53. $f = 15, g = 17, m\angle F = 54$
- 54. $m\angle F = 47, m\angle H = 78, g = 31$
- 55. $m\angle G = 56, m\angle H = 67, g = 63$
- 56. $g = 30.7, h = 32.4, m\angle G = 65$

57. **Proof** Write a two-column proof. **(Lesson 4-5)**

**Given:** $JL \parallel KM$

$JK \parallel LM$

**Prove:** $\triangle JKLM \cong \triangle MLK$

State the transversal that forms each pair of angles. Then identify the special name for the angle pair. **(Lesson 3-1)**

- 58. $\angle 3$ and $\angle 11$
- 59. $\angle 6$ and $\angle 7$
- 60. $\angle 8$ and $\angle 10$
- 61. $\angle 12$ and $\angle 16$

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** In the figure, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. Name all pairs of angles for each type indicated. **(To review angles formed by parallel lines and a transversal, see Lesson 3-1.)**

- 62. consecutive interior angles
- 63. alternate interior angles
- 64. corresponding angles
- 65. alternate exterior angles

www.geometryonline.com/self_check_quiz
Angles of Polygons

It is possible to find the interior and exterior measurements along with the sum of the interior angles of any regular polygon with \( n \) number of sides using a spreadsheet.

Example

Design a spreadsheet using the following steps.

- Label the columns as shown in the spreadsheet below.
- Enter the digits 3–10 in the first column.
- The number of triangles formed by diagonals from the same vertex in a polygon is 2 less than the number of sides. Write a formula for Cell B1 to subtract 2 from each number in Cell A1.
- Enter a formula for Cell C1 so the spreadsheet will find the sum of the measures of the interior angles. Remember that the formula is \( S = \frac{(n - 2)180}{n} \).
- Continue to enter formulas so that the indicated computation is performed. Then, copy each formula through Row 9. The final spreadsheet will appear as below.

<table>
<thead>
<tr>
<th>Number of Sides</th>
<th>Number of Triangles</th>
<th>Sum of Measures of Interior Angles</th>
<th>Measure of Each Interior Angle</th>
<th>Measure of Each Exterior Angle</th>
<th>Sum of Measures of Exterior Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>180</td>
<td>60</td>
<td>120</td>
<td>360</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>360</td>
<td>90</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>540</td>
<td>108</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>720</td>
<td>120</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>900</td>
<td>135</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
<td>1080</td>
<td>144</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>8</td>
<td>13</td>
<td>1260</td>
<td>144</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>9</td>
<td>15</td>
<td>1440</td>
<td>144</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>10</td>
<td>17</td>
<td>1620</td>
<td>144</td>
<td>180</td>
<td>360</td>
</tr>
</tbody>
</table>

Exercises

1. Write the formula to find the measure of each interior angle in the polygon.
2. Write the formula to find the sum of the measures of the exterior angles.
3. What is the measure of each interior angle if the number of sides is 1? 2?
4. Is it possible to have values of 1 and 2 for the number of sides? Explain.

For Exercises 5–8, use the spreadsheet.

5. How many triangles are in a polygon with 15 sides?
6. Find the measure of the exterior angle of a polygon with 15 sides.
7. Find the measure of the interior angle of a polygon with 110 sides.
8. If the measure of the exterior angles is 0, find the measure of the interior angles. Is this possible? Explain.
**Vocabulary**
- parallelogram

**What You’ll Learn**
- Recognize and apply properties of the sides and angles of parallelograms.
- Recognize and apply properties of the diagonals of parallelograms.

**How are parallelograms used to represent data?**

The graphic shows the percent of Global 500 companies that use the Internet to find potential employees. The top surfaces of the wedges of cheese are all polygons with a similar shape. However, the size of the polygon changes to reflect the data. What polygon is this?

**SIDES AND ANGLES OF PARALLELOGRAMS**

A quadrilateral with parallel opposite sides is called a **parallelogram**.

**Key Concept**

<table>
<thead>
<tr>
<th>Words</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>A parallelogram is a quadrilateral with both pairs of opposite sides parallel.</td>
<td>$\Box ABCD$</td>
</tr>
</tbody>
</table>

**Example**

This activity will help you make conjectures about the sides and angles of a parallelogram.

**Geometry Activity**

**Properties of Parallelograms**

**Make a model**

**Step 1** Construct two sets of intersecting parallel lines on patty paper. Label the vertices $FGHI$. 

(continued on the next page)
Proof of Theorem 8.4

Write a two-column proof of Theorem 8.4.

**Given:** \(ABCD\)

**Prove:** \(\angle A \equiv \angle C\)

**\(\angle D \equiv \angle B\)**

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (ABCD)</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. (AB \parallel DC), (AD \parallel BC)</td>
<td>2. Definition of parallelogram</td>
</tr>
<tr>
<td>3. (\angle A \text{ and } \angle D \text{ are supplementary.} ) (\angle D \text{ and } \angle C \text{ are supplementary.} ) (\angle C \text{ and } \angle B \text{ are supplementary.} )</td>
<td>3. If parallel lines are cut by a transversal, consecutive interior angles are supplementary</td>
</tr>
<tr>
<td>4. (\angle A \equiv \angle C) (\angle D \equiv \angle B)</td>
<td>4. Supplements of the same angles are congruent</td>
</tr>
</tbody>
</table>

The Geometry Activity leads to four properties of parallelograms.

<table>
<thead>
<tr>
<th>Key Concept</th>
<th>Properties of Parallelograms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Theorem</strong></td>
<td><strong>Example</strong></td>
</tr>
<tr>
<td>8.3</td>
<td>Opposite sides of a parallelogram are congruent.</td>
</tr>
<tr>
<td><strong>Abbreviation:</strong> Opp. sides of (\square) are (\equiv).</td>
<td></td>
</tr>
<tr>
<td>8.4</td>
<td>Opposite angles in a parallelogram are congruent.</td>
</tr>
<tr>
<td><strong>Abbreviation:</strong> Opp. (\angle) of (\square) are (\equiv).</td>
<td></td>
</tr>
<tr>
<td>8.5</td>
<td>Consecutive angles in a parallelogram are supplementary.</td>
</tr>
<tr>
<td><strong>Abbreviation:</strong> Cons. (\angle) in (\square) are suppl.</td>
<td></td>
</tr>
<tr>
<td>8.6</td>
<td>If a parallelogram has one right angle, it has four right angles.</td>
</tr>
<tr>
<td><strong>Abbreviation:</strong> If (\square) has 1 rt. (\angle), it has 4 rt. (\angle).</td>
<td></td>
</tr>
</tbody>
</table>

You will prove Theorems 8.3, 8.5, and 8.6 in Exercises 41, 42, and 43, respectively.
**Example 2** Properties of Parallelograms

**ALGEBRA** Quadrilateral $LMNP$ is a parallelogram.  
Find $m\angle PLM$, $m\angle LMN$, and $d$.

$m\angle MNP = 66 + 42$ or 108  
Angle Addition Theorem

$\angle PLM \cong \angle MNP$  
Opp. $\angle$ of $\square$ are $\cong$.

$m\angle PLM = m\angle MNP$  
Definition of congruent angles

$m\angle PLM = 108$  
Substitution

$m\angle PLM + m\angle LMN = 180$  
Cons. $\angle$ of $\square$ are suppl.

$108 + m\angle LMN = 180$  
Substitution

$m\angle LMN = 72$  
Subtract 108 from each side.

$m\angle PLM \cong \angle MNP$  
Opp. sides of $\square$ are $\cong$.

$m\angle PLM = m\angle MNP$  
Definition of congruent angles

$m\angle PLM = 108$  
Substitution

$m\angle LMN = \frac{180 - 108}{2} = 42$ or 108  
Substitution

**DIAGONALS OF PARALLELOGRAMS**

In parallelogram $JKLM$, $JL$ and $KM$ are diagonals.  
Theorem 8.7 states the relationship between diagonals of a parallelogram.

**Theorem 8.7**

The diagonals of a parallelogram bisect each other.

**Abbreviation:** Diag. of $\square$ bisect each other.

**Example:** $RQ \cong QT$ and $SQ \cong QU$

You will prove Theorem 8.7 in Exercise 44.

**Example 3** Diagonals of a Parallelogram

**Multiple-Choice Test Item**

What are the coordinates of the intersection of the diagonals of parallelogram $ABCD$ with vertices $A(2, 5)$, $B(6, 6)$, $C(4, 0)$, and $D(0, -1)$?

- **A** $(4, 2)$  
- **B** $(4.5, 2)$  
- **C** $(\frac{7}{6}, \frac{-5}{2})$  
- **D** $(3, 2.5)$

**Read the Test Item**

Since the diagonals of a parallelogram bisect each other, the intersection point is the midpoint of $\overline{AC}$ and $\overline{BD}$.

**Solve the Test Item**

Find the midpoint of $\overline{AC}$.

$$
\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + 4}{2}, \frac{5 + 0}{2}\right)
$$

$$
= (3, 2.5)
$$

The coordinates of the intersection of the diagonals of parallelogram $ABCD$ are $(3, 2.5)$. The answer is D.
Theorem 8.8 describes another characteristic of the diagonals of a parallelogram.

**Theorem 8.8**

Each diagonal of a parallelogram separates the parallelogram into two congruent triangles.

**Abbreviation:** Diag. separates \( \square \) into 2 \( \equiv \) \( \triangle \).  

**Example:** \( \triangle ACD \equiv \triangle CAB \)

You will prove Theorem 8.8 in Exercise 45.

**Check for Understanding**

**Concept Check**

1. Describe the characteristics of the sides and angles of a parallelogram.
2. Describe the properties of the diagonals of a parallelogram.
3. **OPEN ENDED** Draw a parallelogram with one side twice as long as another side.

**Guided Practice**

Complete each statement about \( \square QRST \). Justify your answer.

4. \( SV \equiv ? \)
5. \( \triangle VRS \equiv ? \)
6. \( \angle TSR \) is supplementary to \( ? \).

Use \( \square JKLM \) to find each measure or value if \( JK = 2b + 3 \) and \( JM = 3a \).

7. \( m \angle MJK \)
8. \( m \angle JML \)
9. \( m \angle JKL \)
10. \( m \angle KJL \)
11. \( a \)
12. \( b \)

**PROOF** Write the indicated type of proof.

13. two-column

   **Given:** \( \square VZRQ \) and \( \square WQST \)  
   **Prove:** \( \angle Z \equiv \angle T \)

14. paragraph

   **Given:** \( \square XYZR \), \( WZ \equiv WS \)  
   **Prove:** \( \angle XYR \equiv \angle S \)

**Standardized Test Practice**

15. **MULTIPLE CHOICE** Find the coordinates of the intersection of the diagonals of parallelogram \( GHJK \) with vertices \( G(-3, 4), H(1, 1), J(3, -5), \) and \( K(-1, -2) \).

   - \( \text{A} \) (0, 0.5)  
   - \( \text{B} \) (6, -1)  
   - \( \text{C} \) (0, -0.5)  
   - \( \text{D} \) (5, 0)
Complete each statement about \( \square ABCD \).
Justify your answer.

16. \( \angle DAB \equiv ? \)   
17. \( \angle ABD \equiv ? \)
18. \( \overline{AB} \parallel ? \)   
19. \( \overline{BG} \equiv ? \)
20. \( \triangle ABD \equiv ? \)   
21. \( \triangle ACD \equiv ? \)

**ALGEBRA**  
Use \( \square MNPR \) to find each measure or value.

22. \( m \angle MNP \)   
23. \( m \angle NRP \)
24. \( m \angle RNP \)   
25. \( m \angle RMN \)
26. \( m \angle MQN \)   
27. \( m \angle MQR \)
28. \( x \)   
29. \( y \)
30. \( w \)   
31. \( z \)

**DRAWING**  
For Exercises 32 and 33, use the following information.

The frame of a pantograph is a parallelogram.

32. Find \( x \) and \( EG \) if \( EJ = 2x + 1 \) and \( JG = 3x \).
33. Find \( y \) and \( FH \) if \( HJ = \frac{1}{2}y + 2 \) and \( JF = y - \frac{1}{2} \).

34. **DESIGN**  
The chest of drawers shown at the right is called *Side 2*. It was designed by Shiro Kuramata. Describe the properties of parallelograms the artist used to place each drawer pull.

35. **ALGEBRA**  
Parallelogram \( ABCD \) has diagonals \( \overline{AC} \) and \( \overline{DB} \) that intersect at point \( P \). If \( AB = 3a + 18 \), \( AC = 12a \), \( PB = a + 2b \), and \( PD = 3b + 1 \), find \( a \), \( b \), and \( DB \).

36. **ALGEBRA**  
In parallelogram \( ABCD \), \( AB = 2x + 5 \), \( m \angle BAC = 2y \), \( m \angle B = 120 \), \( m \angle CAD = 21 \), and \( CD = 21 \). Find \( x \) and \( y \).

**COORDINATE GEOMETRY**  
For Exercises 37–39, refer to \( \square EFGH \).

37. Use the Distance Formula to verify that the diagonals bisect each other.
38. Determine whether the diagonals of this parallelogram are congruent.
39. Find the slopes of \( \overline{EH} \) and \( \overline{EF} \). Are the consecutive sides perpendicular? Explain.

40. Determine the relationship among \( \overline{ACBX} \), \( \overline{ABYC} \), and \( \overline{ABCZ} \) if \( \triangle XYZ \) is equilateral and \( A, B, \) and \( C \) are midpoints of \( \overline{XZ} \), \( \overline{XY} \), and \( \overline{ZY} \), respectively.

**PROOF**  
Write the indicated type of proof.

41. two-column proof of Theorem 8.3  
42. two-column proof of Theorem 8.5  
43. paragraph proof of Theorem 8.6  
44. paragraph proof of Theorem 8.7  
45. two-column proof of Theorem 8.8
Maintain Your Skills

Find the sum of the measures of the interior angles of each convex polygon.

52. 14-gon 53. 22-gon 54. 17-gon 55. 36-gon

Determine whether the Law of Sines or the Law of Cosines should be used to solve each triangle. Then solve each triangle. Round to the nearest tenth.

56. 57. 58.

Use Pascal’s Triangle for Exercises 59 and 60.

59. Find the sum of the first 30 numbers in the outside diagonal of Pascal’s triangle.
60. Find the sum of the first 70 numbers in the second diagonal.

Getting Ready for the Next Lesson

PREREQUISITE SKILL The vertices of a quadrilateral are \(A(-5, -2), B(-2, 5), C(2, -2),\) and \(D(-1, -9)\). Determine whether each segment is a side or a diagonal of the quadrilateral, and find the slope of each segment.

61. \(AB\) 62. \(BD\) 63. \(CD\)
CONDITIONS FOR A PARALLELOGRAM  By definition, the opposite sides of a parallelogram are parallel. So, if a quadrilateral has each pair of opposite sides parallel it is a parallelogram. Other tests can be used to determine if a quadrilateral is a parallelogram.

Geometry Activity

Testing for a Parallelogram

Model
• Cut two straws to one length and two other straws to a different length.
• Connect the straws by inserting a pipe cleaner in one end of each size of straw to form a quadrilateral like the one shown at the right.
• Shift the sides to form quadrilaterals of different shapes.

Analyze
1. Measure the distance between the opposite sides of the quadrilateral in at least three places. Repeat this process for several figures. What can you conclude about opposite sides?
2. Classify the quadrilaterals that you formed.
3. Compare the measures of pairs of opposite sides.
4. Measure the four angles in several of the quadrilaterals. What relationships do you find?

Make a Conjecture
5. What conditions are necessary to verify that a quadrilateral is a parallelogram?
Write a Proof

Write a paragraph proof for Theorem 8.10

Given: \( \angle A \cong \angle C, \angle B \cong \angle D \)

Prove: \( ABCD \) is a parallelogram.

Paragraph Proof:

Because two points determine a line, we can draw \( \overline{AC} \). We now have two triangles. We know the sum of the angle measures of a triangle is 180, so the sum of the angle measures of two triangles is 360. Therefore, \( m\angle A + m\angle B + m\angle C + m\angle D = 360 \).

Since \( \angle A \cong \angle C \) and \( \angle B \cong \angle D \), \( m\angle A = m\angle C \) and \( m\angle B = m\angle D \). Substitute to find that \( m\angle A + m\angle A + m\angle B + m\angle D = 360 \), or \( 2(m\angle A) + 2(m\angle B) = 360 \).

Dividing each side of the equation by 2 yields \( m\angle A + m\angle B = 180 \). This means that consecutive angles are supplementary and \( \overline{AD} \parallel \overline{BC} \).

Likewise, \( 2m\angle A + 2m\angle D = 360 \), or \( m\angle A + m\angle D = 180 \). These consecutive supplementary angles verify that \( \overline{AB} \parallel \overline{DC} \). Opposite sides are parallel, so \( ABCD \) is a parallelogram.
**Example 3** **Properties of Parallelograms**

Determine whether the quadrilateral is a parallelogram. Justify your answer.

Each pair of opposite angles have the same measure. Therefore, they are congruent. If both pairs of opposite angles are congruent, the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if any one of the following is true.

**Concept Summary**

**Tests for a Parallelogram**

1. Both pairs of opposite sides are parallel. (Definition)
2. Both pairs of opposite sides are congruent. (Theorem 8.9)
3. Both pairs of opposite angles are congruent. (Theorem 8.10)
4. Diagonals bisect each other. (Theorem 8.11)
5. A pair of opposite sides is both parallel and congruent. (Theorem 8.12)

**Example 4** **Find Measures**

**ALGEBRA** Find $x$ and $y$ so that each quadrilateral is a parallelogram.

**a.**

**Diagram:**

Opposite sides of a parallelogram are congruent.

\[
EF \equiv DG \quad \text{Opp. sides of } \square \text{ are } \equiv.
\]

\[
EF = DG \quad \text{Def. of } \equiv \text{ segments}
\]

\[
4y = 6y - 42 \quad \text{Substitution}
\]

\[
-2y = -42 \quad \text{Subtract } 6y.
\]

\[
y = 21 \quad \text{Divide by } -2.
\]

So, when $x$ is 12 and $y$ is 21, $DEFG$ is a parallelogram.

**b.**

**Diagram:**

Diagonals in a parallelogram bisect each other.

\[
QT \equiv TS \quad \text{Opp. sides of } \square \text{ are } \equiv.
\]

\[
QT = TS \quad \text{Def. of } \equiv \text{ segments}
\]

\[
5y = 2y + 12 \quad \text{Substitution}
\]

\[
3y = 12 \quad \text{Subtract } 2y.
\]

\[
y = 4 \quad \text{Divide by } 3.
\]

So, when $x = 7$ and $y = 4$, $PQRS$ is a parallelogram.
### Example 5 Use Slope and Distance

#### COORDINATE GEOMETRY
Determine whether the figure with the given vertices is a parallelogram. Use the method indicated.

**a.** \(A(3, 3), B(8, 2), C(6, -1), D(1, 0)\); Slope Formula

If the opposite sides of a quadrilateral are parallel, then it is a parallelogram.

\[
slope \text{ of } AB = \frac{2 - 3}{8 - 3} = -\frac{1}{5}
\]

\[
slope \text{ of } DC = \frac{6 - 1}{-1 - 5} = -\frac{1}{5}
\]

Since opposite sides have the same slope, \(AB \parallel DC\) and \(AD \parallel BC\). Therefore, \(ABCD\) is a parallelogram by definition.

**b.** \(P(5, 3), Q(1, -5), R(-6, -1), S(-2, 7)\); Distance and Slope Formulas

First use the Distance Formula to determine whether the opposite sides are congruent.

\[
PS = \sqrt{(5 - (-2))^2 + (3 - 7)^2} = \sqrt{5^2 + (-4)^2} = \sqrt{25 + 16} = \sqrt{41}
\]

\[
QR = \sqrt{(1 - (-6))^2 + (-5 - (-1))^2} = \sqrt{7^2 + (-4)^2} = \sqrt{49 + 16} = \sqrt{65}
\]

Since \(PS = QR\), \(PS \equiv QR\).

Next, use the Slope Formula to determine whether \(PS \parallel QR\).

\[
slope \text{ of } PS = \frac{-7}{5 - (-2)} = \frac{-7}{7} = -1
\]

\[
slope \text{ of } QR = \frac{-1 - (-1)}{6 - (-6)} = \frac{0}{12} = 0
\]

\(PS\) and \(QR\) have the same slope, so they are parallel. Since one pair of opposite sides is congruent and parallel, \(PQRS\) is a parallelogram.

### Check for Understanding

**Concept Check**

1. List and describe four tests for parallelograms.

2. **OPEN ENDED** Draw a parallelogram. Label the congruent angles.

3. **FIND THE ERROR** Carter and Shaniqua are describing ways to show that a quadrilateral is a parallelogram.

   **Carter**
   
   A quadrilateral is a parallelogram if one pair of opposite sides is congruent and one pair of opposite sides is parallel.

   **Shaniqua**
   
   A quadrilateral is a parallelogram if one pair of opposite sides is congruent and parallel.

   Who is correct? Explain your reasoning.
**Guided Practice**

Determine whether each quadrilateral is a parallelogram. Justify your answer.

4. [Diagram]

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram.

6. 

\[2x - 5 \quad 3x - 18\]
\[2y + 12 \quad 5y\]

7. 

\[
\begin{align*}
(3x - 17)^\circ & \quad (y + 58)^\circ \\
(5y - 6)^\circ & \quad (2x + 24)^\circ
\end{align*}
\]

COORDINATE GEOMETRY Determine whether the figure with the given vertices is a parallelogram. Use the method indicated.

8. $B(0, 0), C(4, 1), D(6, 5), E(2, 4)$; Slope Formula

9. $A(-4, 0), B(3, 1), C(1, 4), D(-6, 3)$; Distance and Slope Formulas

10. $E(-4, -3), F(4, -1), G(2, 3), H(-6, 2)$; Midpoint Formula

11. **PROOF** Write a two-column proof to prove that $PQRS$ is a parallelogram given that $\overline{PT} \parallel \overline{TR}$ and $\angle TSP \equiv \angle TQR$.

**Application**

12. **TANGRAMS** A tangram set consists of seven pieces: a small square, two small congruent right triangles, two large congruent right triangles, a medium-sized right triangle, and a quadrilateral. How can you determine the shape of the quadrilateral? Explain.

---

**Practice and Apply**

Determine whether each quadrilateral is a parallelogram. Justify your answer.

13. [Diagram]

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram.

19. 

\[2x \quad 5x - 18\]
\[96 - y \quad 3y\]

20. 

\[8y - 36 \quad 2x + 3\]
\[4y \quad 5x\]

21. 

\[y + 2x \quad 3y + 2x\]
\[5y - 2x \quad 4\]

22. 

\[\frac{25x}{40} \quad \frac{10y}{100}\]
\[\frac{1}{2}y^\circ \quad x^\circ\]

23. 

\[
\begin{align*}
(x - 12)^\circ & \quad (3y - 4)^\circ \\
(3y - 8)^\circ & \quad (4x - 8)^\circ
\end{align*}
\]

24. 

\[
\begin{align*}
4y & \quad 3y + 4 \\
\frac{3}{5}x & \quad \frac{2}{5}x
\end{align*}
\]
COORDINATE GEOMETRY  Determine whether a figure with the given vertices is a parallelogram. Use the method indicated.

25. B(−6, −3), C(−2, −3), E(4, 4), G(−4, 4); Slope Formula
26. Q(−3, −6), R(2, 2), S(−1, 6), T(−5, 2); Slope Formula
27. A(−5, −4), B(3, −2), C(4, 4), D(−4, 2); Distance Formula
28. W(−6, −5), X(−1, −4), Y(0, −1), Z(−5, −2); Midpoint Formula
29. G(−2, 8), H(4, 4), J(6, −3), K(−1, −7); Distance and Slope Formulas
30. H(5, 6), J(9, 0), K(8, −5), L(3, −2); Distance Formula
31. S(−1, 9), T(3, 8), V(6, 2), W(2, 3); Midpoint Formula
32. C(−7, 3), D(−3, 2), F(0, −4), G(−4, −3); Distance and Slope Formulas

33. Quadrilateral MNPR has vertices M(−6, 6), N(1, −1), P(−2, −4), and R(−5, −2). Determine how to move one vertex to make MNPR a parallelogram.

34. Quadrilateral QSTW has vertices Q(−3, 3), S(4, 1), T(1, 2), and W(5, 1). Determine how to move one vertex to make QSTW a parallelogram.

COORDINATE GEOMETRY  The coordinates of three of the vertices of a parallelogram are given. Find the possible coordinates for the fourth vertex.

35. A(1, 4), B(7, 5), and C(4, −1).
36. Q(−2, 2), R(1, 1), and S(−1, −1).

37. STORAGE  Songan purchased an expandable hat rack that has 11 pegs. In the figure, H is the midpoint of KM and JL. What type of figure isJKLM? Explain.

38. METEOROLOGY  To show the center of a storm, television stations superimpose a “watchbox” over the weather map. Describe how you know that the watchbox is a parallelogram.

Online Research Data Update  Each hurricane is assigned a name as the storm develops. What is the name of the most recent hurricane or tropical storm in the Atlantic or Pacific Oceans? Visit www.geometryonline.com/data_update to learn more.

PROOF  Write a two-column proof of each theorem.

39. Theorem 8.9  40. Theorem 8.11  41. Theorem 8.12

42. Li-Cheng claims she invented a new geometry theorem. A diagonal of a parallelogram bisects its angles. Determine whether this theorem is true. Find an example or counterexample.

43. CRITICAL THINKING  Write a proof to prove that FDCA is a parallelogram if ABCDEF is a regular hexagon.

44. WRITING IN MATH  Answer the question that was posed at the beginning of the lesson.

How are parallelograms used in architecture?

Include the following in your answer:

• the information needed to prove that the roof of the covered bridge is a parallelogram, and
• another example of parallelograms used in architecture.
1. The measure of an interior angle of a regular polygon is 147°. Find the number of sides in the polygon. (Lesson 8-1)

2. Use $\square WXYZ$ to find each measure. (Lesson 8-2)

3. $m\angle XYZ = \_\_\_\_\_\_\_\_\_.

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram. (Lesson 8-3)

4. $3x + 9 \degree$, $(3y + 36)\degree$

5. $(6y - 57)\degree$, $(5x - 19)\degree$

45. A parallelogram has vertices at $(-2, 2), (1, -6)$, and $(8, 2)$. Which ordered pair could represent the fourth vertex?

A. $(5, 6)$ B. $(11, -6)$ C. $(14, 3)$ D. $(8, -8)$

46. ALGEBRA Find the distance between $X(5, 7)$ and $Y(-3, -4)$.

A. $\sqrt{19}$ B. $3\sqrt{15}$ C. $\sqrt{185}$ D. $5\sqrt{29}$

Maintain Your Skills

Mixed Review Use $\square NQRM$ to find each measure or value. (Lesson 8-2)

47. $w$

48. $x$

49. $NQ$

50. $QR$

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon. (Lesson 8-1)

51. 135

52. 144

53. 168

54. 162

55. 175

56. 175.5

Find $x$ and $y$. (Lesson 7-3)

57. $x^2 = 12$

58. $y = 12$

59. $x = 32$, $y = 60$

Getting Ready for the Next Lesson PREREQUISITE SKILL Use slope to determine whether $\overline{AB}$ and $\overline{BC}$ are perpendicular or not perpendicular. (To review slope and perpendicularity, see Lesson 3-3.)

60. $A(2, 5), B(6, 3), C(8, 7)$

61. $A(-1, 2), B(0, 7), C(4, 1)$

62. $A(0, 4), B(5, 7), C(8, 3)$

63. $A(-2, -5), B(1, -3), C(-1, 0)$

Practice Quiz 1 Lessons 8-1 through 8-3

1. The measure of an interior angle of a regular polygon is $147\frac{3}{11}$°. Find the number of sides in the polygon. (Lesson 8-1)

Use $\square WXYZ$ to find each measure. (Lesson 8-2)

2. $WZ = \_\_\_\_\_\_\_\_\_\_.$

3. $m\angle XYZ = \_\_\_\_\_\_\_\_.$

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram. (Lesson 8-3)

4. $(3x + 9)\degree$, $(3y + 36)\degree$

5. $(6y - 57)\degree$, $(5x - 19)\degree$

46. ALGEBRA Find the distance between $X(5, 7)$ and $Y(-3, -4)$.

A. $\sqrt{19}$ B. $3\sqrt{15}$ C. $\sqrt{185}$ D. $5\sqrt{29}$

Maintain Your Skills

Mixed Review Use $\square NQRM$ to find each measure or value. (Lesson 8-2)

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51. 135

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56. 175.5

Find $x$ and $y$. (Lesson 7-3)

57. $x^2 = 12$

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Getting Ready for the Next Lesson PREREQUISITE SKILL Use slope to determine whether $\overline{AB}$ and $\overline{BC}$ are perpendicular or not perpendicular. (To review slope and perpendicularity, see Lesson 3-3.)

60. $A(2, 5), B(6, 3), C(8, 7)$

61. $A(-1, 2), B(0, 7), C(4, 1)$

62. $A(0, 4), B(5, 7), C(8, 3)$

63. $A(-2, -5), B(1, -3), C(-1, 0)$

Practice Quiz 1 Lessons 8-1 through 8-3

1. The measure of an interior angle of a regular polygon is $147\frac{3}{11}$°. Find the number of sides in the polygon. (Lesson 8-1)

Use $\square WXYZ$ to find each measure. (Lesson 8-2)

2. $WZ = \_\_\_\_\_\_\_\_\_\_.$

3. $m\angle XYZ = \_\_\_\_\_\_\_\_.$

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram. (Lesson 8-3)

4. $(3x + 9)\degree$, $(3y + 36)\degree$

5. $(6y - 57)\degree$, $(5x - 19)\degree$
8-4 Rectangles

**What You’ll Learn**
- Recognize and apply properties of rectangles.
- Determine whether parallelograms are rectangles.

**Vocabulary**
- rectangle

**How are rectangles used in tennis?**

Many sports are played on fields marked by parallel lines. A tennis court has parallel lines at half-court for each player. Parallel lines divide the court for singles and doubles play. The service box is marked by perpendicular lines.

**Properties of Rectangles**

A rectangle is a quadrilateral with four right angles. Since both pairs of opposite angles are congruent, it follows that it is a special type of parallelogram. Thus, a rectangle has all the properties of a parallelogram. Because the right angles make a rectangle a rigid figure, the diagonals are also congruent.

**Theorem 8.13**

If a parallelogram is a rectangle, then the diagonals are congruent.

**Abbreviation:** If $\square$ is rectangle, diag. are $\cong$.

$\overline{AC} \cong \overline{BD}$

You will prove Theorem 8.13 in Exercise 40.

If a quadrilateral is a rectangle, then the following properties are true.

**Key Concept**

**Rectangle**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Opposite sides are congruent and parallel.</td>
<td>$\overline{AB} \cong \overline{DC}$, $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \cong \overline{AD}$, $\overline{BC} \parallel \overline{AD}$</td>
</tr>
<tr>
<td>2. Opposite angles are congruent.</td>
<td>$\angle A \cong \angle C$, $\angle B \cong \angle D$</td>
</tr>
<tr>
<td>3. Consecutive angles are supplementary.</td>
<td>$m\angle A + m\angle B = 180$, $m\angle B + m\angle C = 180$, $m\angle C + m\angle D = 180$, $m\angle D + m\angle A = 180$</td>
</tr>
<tr>
<td>4. Diagonals are congruent and bisect each other.</td>
<td>$\overline{AC}$ and $\overline{BD}$ bisect each other. $\overline{AC} \equiv \overline{BD}$</td>
</tr>
<tr>
<td>5. All four angles are right angles.</td>
<td>$m\angle DAB = m\angle BCD = m\angle ABC = m\angle ADC = 90$</td>
</tr>
</tbody>
</table>
**Example 1** Diagonals of a Rectangle

**ALGEBRA** Quadrilateral $MNOP$ is a rectangle. If $MO = 6x + 14$ and $PN = 9x + 5$, find $x$.

The diagonals of a rectangle are congruent, so $MO \cong PN$.

$$MO \cong PN \quad \text{Diagonals of a rectangle are } \cong.$$  
$$MO = PN \quad \text{Definition of congruent segments}$$

$$6x + 14 = 9x + 5 \quad \text{Substitution}$$

$$14 = 3x + 5 \quad \text{Subtract 6x from each side.}$$

$$9 = 3x \quad \text{Subtract 5 from each side.}$$

$$3 = x \quad \text{Divide each side by 3.}$$

Rectangles can be constructed using perpendicular lines.

**Construction**

**Rectangle**

1. Use a straightedge to draw line $\ell$. Label a point $P$ on $\ell$. Place the point at $P$ and mark off a segment on $m$. Now construct lines perpendicular to $\ell$ through $P$ and through $Q$. Label them $m$ and $n$.

2. Place the compass point at $P$ and mark off a segment on $m$. Using the same compass setting, place the compass at $Q$ and mark a segment on $n$. Label these points $R$ and $S$. Draw $RS$.

3. Locate the compass setting that represents $PR$ and compare to the setting for $QS$. The measures should be the same.

**Example 2** Angles of a Rectangle

**ALGEBRA** Quadrilateral $ABCD$ is a rectangle.

a. Find $x$.

$\angle DAB$ is a right angle, so $m\angle DAB = 90$.  

$m\angle DAC + m\angle BAC = m\angle DAB$  

$$4x + 5 + 9x + 20 = 90 \quad \text{Angle Addition Theorem}$$

$$13x + 25 = 90 \quad \text{Substitution}$$

$$13x = 65 \quad \text{Simplify.}$$

$$x = 5 \quad \text{Subtract 25 from each side.}$$

www.geometryonline.com/extra_examples
b. Find $y$.

Since a rectangle is a parallelogram, opposite sides are parallel. So, alternate interior angles are congruent.

\[
\angle ADB \cong \angle CBD \quad \text{Alternate Interior Angles Theorem}
\]
\[
m\angle ADB = m\angle CBD \quad \text{Definition of } \cong \text{ angles}
\]
\[
y^2 - 1 = 4y + 4 \quad \text{Substitution}
\]
\[
y^2 - 4y - 5 = 0 \quad \text{Subtract } 4y \text{ and } 4 \text{ from each side.}
\]
\[
(y - 5)(y + 1) = 0 \quad \text{Factor.}
\]
\[
y - 5 = 0 \quad y + 1 = 0 \quad \text{Disregard } y = -1 \text{ because it yields angle measures of } 0.
\]
\[
y = 5 \quad y = -1
\]

PROVE THAT PARALLELOGRAMS ARE RECTANGLES

The converse of Theorem 8.13 is also true.

**Theorem 8.14**

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

**Abbreviation:** If diagonals of $\square$ are $\cong$, $\square$ is a rectangle.

\[AB \cong DC\]

You will prove Theorem 8.14 in Exercise 41.

**Example 3**

**Diagonals of a Parallelogram**

**WINDOWS**

Trent is building a tree house for his younger brother. He has measured the window opening to be sure that the opposite sides are congruent. He measures the diagonals to make sure that they are congruent. This is called *squaring* the frame. How does he know that the corners are $90^\circ$ angles?

First draw a diagram and label the vertices. We know that $WX \cong ZY$, $XY \cong WZ$, and $WY \cong XZ$.

Because $WX \cong ZY$ and $XY \cong WZ$, $WXYZ$ is a parallelogram.

$XZ$ and $WY$ are diagonals and they are congruent. A parallelogram with congruent diagonals is a rectangle. So, the corners are $90^\circ$ angles.

**Example 4**

**Rectangle on a Coordinate Plane**

**COORDINATE GEOMETRY** Quadrilateral $FGHJ$ has vertices $F(-4, -1), G(-2, -5), H(4, -2)$, and $J(2, 2)$. Determine whether $FGHJ$ is a rectangle.

**Method 1:** Use the Slope Formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to see if consecutive sides are perpendicular.

slope of $FJ = \frac{2 - (-1)}{2 - (-4)}$ or $\frac{1}{2}$
Concept Check
1. How can you determine whether a parallelogram is a rectangle?

2. OPEN ENDED  Draw two congruent right triangles with a common hypotenuse. Do the legs form a rectangle?

3. FIND THE ERROR   McKenna and Consuelo are defining a rectangle for an assignment.

   **McKenna**
   
   A rectangle is a parallelogram with one right angle.

   **Consuelo**
   
   A rectangle has a pair of parallel opposite sides and a right angle.

Who is correct? Explain.
4. **ALGEBRA**  
**ABCD** is a rectangle.  
If \( AC = 30 - x \) and \( BD = 4x - 60 \), find \( x \).

![Diagram of a rectangle ABCD with diagonals AC and BD]

5. **ALGEBRA**  
**MNQR** is a rectangle.  
If \( NR = 2x + 10 \) and \( NP = 2x - 30 \), find \( MP \).

![Diagram of a rectangle MNQR with diagonals MR and NP]

**ALGEBRA**  
Quadrilateral **QRST** is a rectangle.  
Find each measure or value.

6. \( x \)
7. \( m \angle RPS \)

8. **COORDINATE GEOMETRY**  
Quadrilateral **EFGH** has vertices \( E(-4, -3) \), \( F(3, -1) \), \( G(2, 3) \), and \( H(-5, 1) \). Determine whether **EFGH** is a rectangle.

9. **FRAMING**  
Mrs. Walker has a rectangular picture that is 12 inches by 48 inches. Because this is not a standard size, a special frame must be built. What can the framer do to guarantee that the frame is a rectangle? Justify your reasoning.

**Practice and Apply**

10. If \( NQ = 5x - 3 \) and \( QM = 4x + 6 \), find \( NK \).
11. If \( NQ = 2x + 3 \) and \( QK = 5x - 9 \), find \( JQ \).
12. If \( NM = 8x - 14 \) and \( JK = x^2 + 1 \), find \( JK \).
13. If \( m \angle NJM = 2x - 3 \) and \( m \angle KJM = x + 5 \), find \( x \).
14. If \( m \angle NKM = x^2 + 4 \) and \( m \angle KNM = x + 30 \), find \( m \angle JKN \).
15. If \( m \angle JKN = 2x^2 + 2 \) and \( m \angle NKM = 14x \), find \( x \).

**WXYZ** is a rectangle. Find each measure if \( m \angle 1 = 30 \).

16. \( m \angle 1 \)
17. \( m \angle 2 \)
18. \( m \angle 3 \)
19. \( m \angle 4 \)
20. \( m \angle 5 \)
21. \( m \angle 6 \)
22. \( m \angle 7 \)
23. \( m \angle 8 \)
24. \( m \angle 9 \)

25. **PATIOS**  
A contractor has been hired to pour a rectangular concrete patio. How can he be sure that the frame in which to pour the concrete is rectangular?

26. **TELEVISION**  
Television screens are measured on the diagonal. What is the measure of the diagonal of this screen?
Lesson 8-4

Rectangles

COORDINATE GEOMETRY  Determine whether $DFGH$ is a rectangle given each set of vertices. Justify your answer.
27. $D(9, -1)$, $F(9, 5)$, $G(-6, 5)$, $H(-6, 1)$
28. $D(6, 2)$, $F(8, -1)$, $G(10, 6)$, $H(12, 3)$
29. $D(-4, -3)$, $F(-5, 8)$, $G(6, 9)$, $H(7, -2)$

COORDINATE GEOMETRY  The vertices of $WXYZ$ are $W(2, 4)$, $X(-2, 0)$, $Y(-1, -7)$, and $Z(9, 3)$.
30. Find $WY$ and $XZ$.
31. Find the coordinates of the midpoints of $WX$ and $YZ$.
32. Is $WXYZ$ a rectangle? Explain.

COORDINATE GEOMETRY  The vertices of parallelogram $ABCD$ are $A(-4, -4)$, $B(2, -1)$, $C(0, 3)$, and $D(-6, 0)$.
33. Determine whether $ABCD$ is a rectangle.
34. If $ABCD$ is a rectangle and $E$, $F$, $G$, and $H$ are midpoints of its sides, what can you conclude about $EFGH$?

MINIATURE GOLF  The windmill section of a miniature golf course will be a rectangle 10 feet long and 6 feet wide. Suppose the contractor placed stakes and strings to mark the boundaries with the corners at $A$, $B$, $C$, and $D$. The contractor measured $BD$ and $AC$ and found that $AC > BD$. Describe where to move the stakes $L$ and $K$ to make $ABCD$ a rectangle. Explain.

GOLDEN RECTANGLES  For Exercises 36 and 37, use the following information.
Many artists have used golden rectangles in their work. In a golden rectangle, the ratio of the length to the width is about 1.618. This ratio is known as the golden ratio.
36. A rectangle has dimensions of 19.42 feet and 12.01 feet. Determine if the rectangle is a golden rectangle. Then find the length of the diagonal.
37. RESEARCH  Use the Internet or other sources to find examples of golden rectangles.
38. What are the minimal requirements to justify that a parallelogram is a rectangle?
39. Draw a counterexample to the statement If the diagonals are congruent, the quadrilateral is a rectangle.

PROOF  Write a two-column proof.
40. Theorem 8.13
41. Theorem 8.14
42. Given: $PQST$ is a rectangle. $QR \equiv VT$
Prove: $PR \equiv VS$

44. CRITICAL THINKING  Using four of the twelve points as corners, how many rectangles can be drawn?
SPHERICAL GEOMETRY  The figure shows a Saccheri quadrilateral on a sphere. Note that it has four sides with $CT \perp TR$, $AR \perp TR$, and $CT \equiv AR$.

45. Is $CT$ parallel to $AR$? Explain.

46. How does $AC$ compare to $TR$?

47. Can a rectangle exist in spherical geometry? Explain.

48. **WRITING IN MATH**  Answer the question that was posed at the beginning of the lesson.

**How are rectangles used in tennis?**

Include the following in your answer:
- the number of rectangles on one side of a tennis court,
- a method to ensure the lines on the court are parallel

49. In the figure, $\overline{AB} \parallel \overline{CE}$. If $DA = 6$, what is $DB$?

   - A 6
   - B 7
   - C 8
   - D 9

50. **ALGEBRA**  A rectangular playground is surrounded by an 80-foot long fence. One side of the playground is 10 feet longer than the other. Which of the following equations could be used to find $s$, the shorter side of the playground?

   - A $10s + s = 80$
   - B $4s + 10 = 80$
   - C $s(s + 10) = 80$
   - D $2(s + 10) + 2s = 80$

---

**Maintain Your Skills**

**Mixed Review**

51. **TEXTILE ARTS**  The Navajo people are well known for their skill in weaving. The design at the right, known as the Eye-Dazzler, became popular with Navajo weavers in the 1880s. How many parallelograms, not including rectangles, are in the pattern?  (Lesson 8-3)

For Exercises 52–57, use $\square ABCD$. Find each measure or value.  (Lesson 8-2)

52. $m\angle AFD$
53. $m\angle CDF$
54. $m\angle FBC$
55. $m\angle BCF$
56. $y$
57. $x$

Find the measure of the altitude of each triangle.  (Lesson 7-2)

58.

59.

60.

---

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL**  Find the distance between each pair of points.

(To review the Distance Formula, see Lesson 1-4.)

61. $(1, -2), (-3, 1)$
62. $(-5, 9), (5, 12)$
63. $(1, 4), (22, 24)$
### Key Concept

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8.15</strong></td>
<td>The diagonals of a rhombus are perpendicular.</td>
</tr>
<tr>
<td><strong>8.16</strong></td>
<td>If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus. (Converse of Theorem 8.15)</td>
</tr>
<tr>
<td><strong>8.17</strong></td>
<td>Each diagonal of a rhombus bisects a pair of opposite angles.</td>
</tr>
</tbody>
</table>

#### Study Tip

**Proof**

Since a rhombus has four congruent sides, one diagonal separates the rhombus into two congruent isosceles triangles. Drawing two diagonals separates the rhombus into four congruent right triangles.

#### Example 1

**Proof of Theorem 8.15**

**Given:** $PQRS$ is a rhombus.

**Prove:** $PR \perp SQ$

**Proof:**

By the definition of a rhombus, $\overline{PQ} \equiv \overline{QR} \equiv \overline{RS} \equiv \overline{PS}$. A rhombus is a parallelogram and the diagonals of a parallelogram bisect each other, so $QS$ bisects $PR$ at $T$. Thus, $\overline{PT} \equiv \overline{RT}$, $\overline{QT} \equiv \overline{QT}$ because congruence of segments is reflexive. Thus, $\triangle PQT \equiv \triangle RQT$ by SSS. $\angle QTP \equiv \angle QTR$ by CPCTC. $\angle QTP$ and $\angle QTR$ also form a linear pair. Two congruent angles that form a linear pair are right angles. $\angle QTP$ is a right angle, so $\overline{PR} \perp \overline{SQ}$ by the definition of perpendicular lines.
**Example 2** Measures of a Rhombus

**ALGEBRA** Use rhombus \(QRST\) and the given information to find the value of each variable.

**a.** Find \(y\) if \(m\angle 3 = y^2 - 31\).

\[
m\angle 3 = 90 \quad \text{The diagonals of a rhombus are perpendicular.}
\]

\[
y^2 - 31 = 90 \quad \text{Substitution}
\]

\[
y^2 = 121 \quad \text{Add 31 to each side.}
\]

\[
y = \pm 11 \quad \text{Take the square root of each side.}
\]

The value of \(y\) can be 11 or \(-11\).

**b.** Find \(m\angle TQS\) if \(m\angle RST = 56\).

\[
m\angle TQR = m\angle RST \quad \text{Opposite angles are congruent.}
\]

\[
m\angle TQR = 56 \quad \text{Substitution}
\]

The diagonals of a rhombus bisect the angles. So, \(m\angle TQS = \frac{1}{2}(56) = 28\).

---

**PROPERTIES OF SQUARES** If a quadrilateral is both a rhombus and a rectangle, then it is a **square**. All of the properties of parallelograms and rectangles can be applied to squares.

**Example 3** Squares

**COORDINATE GEOMETRY** Determine whether parallelogram \(ABCD\) is a rhombus, a rectangle, or a square. List all that apply. Explain.

**Explore** Plot the vertices on a coordinate plane.

**Plan** If the diagonals are perpendicular, then \(ABCD\) is either a rhombus or a square. The diagonals of a rectangle are congruent. If the diagonals are congruent and perpendicular, then \(ABCD\) is a square.

**Solve** Use the Distance Formula to compare the lengths of the diagonals.

\[
DB = \sqrt{(3 - (-3))^2 + (-1 - 1)^2} \quad AC = \sqrt{(1 + 1)^2 + (3 + 3)^2}
\]

\[
= \sqrt{36 + 4} \quad = \sqrt{4 + 36}
\]

\[
= \sqrt{40} \quad = \sqrt{40}
\]

Use slope to determine whether the diagonals are perpendicular.

\[
slope of \overline{DB} = \frac{1 - (-1)}{-3 - 3} = \frac{2}{-6} = -\frac{1}{3} \quad slope of \overline{AC} = \frac{-3 - 3}{-1 - 1} = \frac{-6}{-2} = 3
\]

Since the slope of \(\overline{AC}\) is the negative reciprocal of the slope of \(\overline{DB}\), the diagonals are perpendicular. The lengths of \(\overline{DB}\) and \(\overline{AC}\) are the same so the diagonals are congruent. \(ABCD\) is a rhombus, a rectangle, and a square.

**Examine** The diagonals are congruent and perpendicular so \(ABCD\) must be a square. You can verify that \(ABCD\) is a rhombus by finding \(AB, BC, CD,\) and \(AD\). Then see if two consecutive segments are perpendicular.
### Construction

**Rhombus**

1. Draw any segment $\overline{AD}$. Place the compass point at $A$, open to the width of $\overline{AD}$, and draw an arc above $\overline{AD}$.

2. Label any point on the arc as $B$. Using the same setting, place the compass at $B$, and draw an arc to the right of $B$.

3. Then place the compass at point $D$, and draw an arc to intersect the arc drawn from point $B$. Label the point of intersection $C$.

4. Use a straightedge to draw $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$.

**Conclusion:** Since all of the sides are congruent, quadrilateral $ABCD$ is a rhombus.

---

### Example 4 Diagonals of a Square

**Baseball**  The infield of a baseball diamond is a square, as shown at the right. Is the pitcher’s mound located in the center of the infield? Explain.

Since a square is a parallelogram, the diagonals bisect each other. Since a square is a rhombus, the diagonals are congruent. Therefore, the distance from first base to third base is equal to the distance between home plate and second base. Thus, the distance from home plate to the center of the infield is $127 \text{ ft } 3 \frac{3}{8} \text{ in.}$ divided by 2 or $63 \text{ ft } 1 \frac{11}{16} \text{ in.}$ This distance is longer than the distance from home plate to the pitcher’s mound so the pitcher’s mound is not located in the center of the field. It is about 3 feet closer to home.

If a quadrilateral is a rhombus or a square, then the following properties are true.

---

### Concept Summary

**Rhombi**

1. A rhombus has all the properties of a parallelogram.
2. All sides are congruent.
3. Diagonals are perpendicular.
4. Diagonals bisect the angles of the rhombus.

**Squares**

1. A square has all the properties of a parallelogram.
2. A square has all the properties of a rectangle.
3. A square has all the properties of a rhombus.
1. Draw a diagram to demonstrate the relationship among parallelograms, rectangles, rhombi, and squares.

2. OPEN ENDED Draw a quadrilateral that has the characteristics of a rectangle, a rhombus, and a square.

3. Explain the difference between a square and a rectangle.

ALGEBRA In rhombus $ABCD$, $AB = 2x + 3$ and $BC = 5x$.

4. Find $x$.  
5. Find $AD$.
6. Find $m\angle AEB$.  
7. Find $m\angle BCD$ if $m\angle ABC = 83.2$.

COORDINATE GEOMETRY Given each set of vertices, determine whether $\square MNPQ$ is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.

8. $M(0, 3), N(-3, 0), P(0, -3), Q(3, 0)$
9. $M(-4, 0), N(-3, 3), P(2, 2), Q(1, -1)$

10. PROOF Write a two-column proof.

   Given: $\triangle KGH, \triangle HKJ, \triangle GHJ$, and $\triangle JKG$ are isosceles.
   Prove: $GHJK$ is a rhombus.

Application

11. REMODELING The Steiner family is remodeling their kitchen. Each side of the floor measures 10 feet. What other measurements should be made to determine whether the floor is a square?

Practice and Apply

In rhombus $ABCD$, $m\angle DAB = 2m\angle ADC$ and $CB = 6$.

12. Find $m\angle ACD$.  
13. Find $m\angle DAB$.
14. Find $DA$.
15. Find $m\angle ADB$.

ALGEBRA Use rhombus $WXYZ$ with $m\angle WYZ = 53$, $VW = 3$, $XV = 2a - 2$, and $ZV = \frac{5a + 1}{4}$.

16. Find $m\angle YZV$.
17. Find $m\angle XYW$.
18. Find $XZ$.
19. Find $XW$.
20. $E(1, 10), F(-4, 0), G(7, 2), H(12, 12)$
21. $E(-7, 3), F(-2, 3), G(1, 7), H(-4, 7)$
22. $E(1, 5), F(6, 5), G(6, 10), H(1, 10)$
23. $E(-2, -1), F(-4, 3), G(1, 5), H(3, 1)$

COORDINATE GEOMETRY Given each set of vertices, determine whether $\square EFGH$ is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.

For Exercises See Examples
12–19 2
20–23 3
24–36 4
37–42 1

Extra Practice See page 770.
CONSTRUCTION  Construct each figure using a compass and straightedge.

24. a square with one side 3 centimeters long
25. a square with a diagonal 5 centimeters long

Use the Venn diagram to determine whether each statement is always, sometimes, or never true.

26. A parallelogram is a square.
27. A square is a rhombus.
28. A rectangle is a parallelogram.
29. A rhombus is a rectangle.
30. A rhombus is a square.
31. A square is a rectangle.

32. DESIGN  Otto Prutscher designed the plant stand at the left in 1903. The base is a square, and the base of each of the five boxes is also a square. Suppose each smaller box is one half as wide as the base. Use the information at the left to find the dimensions of one of the smaller boxes.

33. PERIMETER  The diagonals of a rhombus are 12 centimeters and 16 centimeters long. Find the perimeter of the rhombus.

34. ART  This piece of art is Dorothea Rockburne’s Egyptian Painting: Scribe. The diagram shows three of the shapes shown in the piece. Use a ruler or a protractor to determine which type of quadrilateral is represented by each figure.

PROOF  Write a paragraph proof for each theorem.

35. Theorem 8.16
36. Theorem 8.17

SQUASH  For Exercises 37 and 38, use the diagram of the court for squash, a game similar to racquetball and tennis.

37. The diagram labels the diagonal as 11,665 millimeters. Is this correct? Explain.
38. The service boxes are squares. Find the length of the diagonal.
39. **FLAGS** Study the flags shown below. Use a ruler and protractor to determine if any of the flags contain parallelograms, rectangles, rhombi, or squares.

![Flags](image)

- Denmark
- St. Vincent and The Grenadines
- Trinidad and Tobago

**PROOF** Write a two-column proof.

40. **Given:** \( \triangle WZY \cong \triangle WXY, \triangle WZY \) and \( \triangle XYZ \) are isosceles.

**Prove:** \( WXYZ \) is a rhombus.

![Proof Diagram]

41. **Given:** \( \triangle TPX \cong \triangle QPX \cong \triangle QRX \cong \triangle TRX \)

**Prove:** \( TPQR \) is a rhombus.

![Proof Diagram]

42. **Given:** \( \triangle LGK \cong \triangle MJK \)

**Prove:** \( GHJK \) is a parallelogram.

![Proof Diagram]

43. **Given:** \( QRST \) and \( QRTV \) are rhombi.

**Prove:** \( \triangle QRT \) is equilateral.

![Proof Diagram]

44. **CRITICAL THINKING**

The pattern at the right is a series of rhombi that continue to form a hexagon that increases in size. Copy and complete the table.

<table>
<thead>
<tr>
<th>Hexagon</th>
<th>Number of rhombi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td></td>
</tr>
</tbody>
</table>

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

**How can you ride a bicycle with square wheels?**

Include the following in your answer:
- difference between squares and rhombi, and
- how nonsquare rhombus-shaped wheels would work with the curved road.
46. Points $A, B, C,$ and $D$ are on a square. The area of the square is 36 square units. Which of the following statements is true?
   - (A) The perimeter of rectangle $ABCD$ is greater than 24 units.
   - (B) The perimeter of rectangle $ABCD$ is less than 24 units.
   - (C) The perimeter of rectangle $ABCD$ is equal to 24 units.
   - (D) The perimeter of rectangle $ABCD$ cannot be determined from the information given.

47. **ALGEBRA** For all integers $x \neq 2$, let $<x> = \frac{1 + x}{x - 2}$. Which of the following has the greatest value?
   - (A) $<0>$
   - (B) $<1>$
   - (C) $<3>$
   - (D) $<4>$

---

**Maintain Your Skills**

**Mixed Review**

**ALGEBRA** Use rectangle $LMNP$, parallelogram $LKMJ$, and the given information to solve each problem. *(Lesson 8-4)*

48. If $LN = 10$, $LJ = 2x + 1$, and $PJ = 3x - 1$, find $x$.
49. If $m\angle PLK = 110$, find $m\angle LKM$.
50. If $m\angle MJN = 35$, find $m\angle MPN$.
51. If $MK = 6x$, $KL = 3x + 2y$, and $JN = 14 - x$, find $x$ and $y$.
52. If $m\angle LMP = m\angle PMN$, find $m\angle P\angle L$.

**COORDINATE GEOMETRY** Determine whether the coordinates of the vertices of the quadrilateral form a parallelogram. Use the method indicated. *(Lesson 8-3)*

53. $P(0, 2), Q(6, 4), R(4, 0), S(-2, -2)$; Distance Formula
54. $F(1, -1), G(-4, 1), H(-3, 4), J(2, 1)$; Distance Formula
55. $K(-3, -7), L(3, 2), M(1, 7), N(-3, 1)$; Slope Formula
56. $A(-4, -1), B(-2, -5), C(1, 7), D(3, 3)$; Slope Formula

Refer to $\triangle PQS$. *(Lesson 6-4)*

57. If $RT = 16$, $QP = 24$, and $ST = 9$, find $PS$.
58. If $PT = y - 3$, $PS = y + 2$, $RS = 12$, and $QS = 16$, solve for $y$.
59. If $RT = 15$, $QP = 21$, and $PT = 8$, find $TS$.

Refer to the figure. *(Lesson 4-6)*

60. If $\overline{AG} \cong \overline{AC}$, name two congruent angles.
61. If $\overline{AJ} \cong \overline{AH}$, name two congruent angles.
62. If $\angle AFD \cong \angle ADF$, name two congruent segments.
63. If $\angle AKB \cong \angle ABK$, name two congruent segments.

---

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Solve each equation. *(To review solving equations, see pages 737 and 738.)*

64. $\frac{1}{2}(8x - 6x - 7) = 5$
65. $\frac{1}{2}(7x + 3x + 1) = 12.5$
66. $\frac{1}{2}(4x + 6 + 2x + 13) = 15.5$
67. $\frac{1}{2}(7x - 2 + 3x + 3) = 25.5$
**Kites**

A *kite* is a quadrilateral with exactly two distinct pairs of adjacent congruent sides. In kite $ABCD$, diagonal $BD$ separates the kite into two congruent triangles. Diagonal $AC$ separates the kite into two noncongruent isosceles triangles.

**Activity**

Construct a kite $QRST$.

1. Draw $RT$.

2. Choose a compass setting greater than $\frac{1}{2} RT$. Place the compass at point $R$ and draw an arc above $RT$. Then without changing the compass setting, move the compass to point $T$ and draw an arc that intersects the first one. Label the intersection point $Q$. Increase the compass setting. Place the compass at $R$ and draw an arc below $RT$. Then, without changing the compass setting, draw an arc from point $T$ to intersect the other arc. Label the intersection point $S$.

3. Draw $QRST$.

**Model**

1. Draw $QS$ in kite $QRST$. Use a protractor to measure the angles formed by the intersection of $QS$ and $RT$.
2. Measure the interior angles of kite $QRST$. Are any congruent?
3. Label the intersection of $QS$ and $RT$ as point $N$. Find the lengths of $QN$, $NS$, $TN$, and $NR$. How are they related?
4. How many pairs of congruent triangles can be found in kite $QRST$?

**Analyze**

6. Use your observations and measurements of kites $QRST$ and $JKLM$ to make conjectures about the angles, sides, and diagonals of kites.
**What You’ll Learn**

- Recognize and apply the properties of trapezoids.
- Solve problems involving the medians of trapezoids.

**Vocabulary**

- trapezoid
- isosceles trapezoid
- median

**How are trapezoids used in architecture?**

The Washington Monument in Washington, D.C., is an obelisk made of white marble. The width of the base is longer than the width at the top. Each face of the monument is an example of a trapezoid.

**PROPERTIES OF TRAPEZOIDS** A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are called bases. The base angles are formed by the base and one of the legs. The nonparallel sides are called legs.

If the legs are congruent, then the trapezoid is an isosceles trapezoid. Theorems 8.18 and 8.19 describe two characteristics of isosceles trapezoids.

**Theorems**

- **8.18** Both pairs of base angles of an isosceles trapezoid are congruent.
- **8.19** The diagonals of an isosceles trapezoid are congruent.

**Example 1 Proof of Theorem 8.19**

Write a flow proof of Theorem 8.19.

**Given:** MNOP is an isosceles trapezoid.

**Prove:** MO ≅ NP

**Proof:**

1. MNOP is an isosceles trapezoid. (Given)
2. ∠MPO ≅ ∠NOP. (Base ∆ of isos. trap. are ≅.
3. MP ≅ NO. (Def. of isos. trapezoid)
4. MO ≅ NP. (CPCTC)
Example 2  Identify Isosceles Trapezoids

**ART**  The sculpture pictured is *Zim Zum I* by Barnett Newman. The walls are connected at right angles. In perspective, the rectangular panels appear to be trapezoids. Use a ruler and protractor to determine if the images of the front panels are isosceles trapezoids. Explain.

The panel on the left is an isosceles trapezoid. The bases are parallel and are different lengths. The legs are not parallel and they are the same length.

The panel on the right is not an isosceles trapezoid. Each side is a different length.

Example 3  Identify Trapezoids

**COORDINATE GEOMETRY**  *JKLM* is a quadrilateral with vertices *J*(−18, −1), *K*(6, 8), *L*(18, 1), and *M*(−18, −26).

a. Verify that *JKLM* is a trapezoid.

A quadrilateral is a trapezoid if exactly one pair of opposite sides are parallel. Use the Slope Formula.

\[
\text{slope of } \overline{JK} = \frac{-1 - 8}{-18 - (-6)} = \frac{-9}{-12} = \frac{3}{4} \\
\text{slope of } \overline{ML} = \frac{1 - (-26)}{18 - (-18)} = \frac{27}{36} = \frac{3}{4}
\]

\[
\text{slope of } \overline{JM} = \frac{-1 - (-26)}{-18 - (-18)} = 1 \\
\text{slope of } \overline{KL} = \frac{1 - (-8)}{18 - (-6)} = -\frac{7}{24}
\]

Exactly one pair of opposite sides are parallel, *JK* and *ML*.

So, *JKLM* is a trapezoid.

b. Determine whether *JKLM* is an isosceles trapezoid. Explain.

First use the Distance Formula to show that the legs are congruent.

\[
\begin{align*}
JKLM & = \sqrt{[-18 - (-18)]^2 + [-1 - (-26)]^2} \\
& = \sqrt{0 + 625} \\
& = \sqrt{625} = 25
\end{align*}
\]

\[
KL = \sqrt{(-6 - 18)^2 + (8 - 1)^2} = \sqrt{576 + 49} = \sqrt{625} = 25
\]

Since the legs are congruent, *JKLM* is an isosceles trapezoid.

**MEDIANS OF TRAPEZIODS**  The segment that joins midpoints of the legs of a trapezoid is the **median**. The median of a trapezoid can also be called a **midsegment**. Recall from Lesson 6-4 that the midsegment of a triangle is the segment joining the midpoints of two sides. The median of a trapezoid has the same properties as the midsegment of a triangle. You can construct the median of a trapezoid using a compass and a straightedge.
Geometry Activity

Median of a Trapezoid

Model
1. Draw trapezoid $WXYZ$ with legs $XY$ and $WZ$.
2. Construct the bisectors of $XY$ and $WZ$. Label the midpoints.
3. Draw $MN$.

Analyze
1. Measure $WX$, $ZY$, and $MN$ to the nearest millimeter.
2. Make a conjecture based on your observations.

The results of the Geometry Activity suggest Theorem 8.20.

Theorem 8.20
The median of a trapezoid is parallel to the bases, and its measure is one-half the sum of the measures of the bases.

Example: $EF = \frac{1}{2}(AB + DC)$

Example 4

Median of a Trapezoid

ALGEBRA $QRST$ is an isosceles trapezoid with median $XY$.

a. Find $TS$ if $QR = 22$ and $XY = 15$.

$XY = \frac{1}{2}(QR + TS)$ \hspace{1cm} \text{Theorem 8.20}

$15 = \frac{1}{2}(22 + TS)$ \hspace{1cm} \text{Substitution}

$30 = 22 + TS$ \hspace{1cm} \text{Multiply each side by 2.}

$8 = TS$ \hspace{1cm} \text{Subtract 22 from each side.}

b. Find $m\angle 1$, $m\angle 2$, $m\angle 3$, and $m\angle 4$ if $m\angle 1 = 4a - 10$ and $m\angle 3 = 3a + 32.5$.

Since $QR \parallel TS$, $\angle 1$ and $\angle 3$ are supplementary. Because this is an isosceles trapezoid, $\angle 1 \equiv \angle 2$ and $\angle 3 \equiv \angle 4$.

$m\angle 1 + m\angle 3 = 180$ \hspace{1cm} \text{Consecutive Interior Angles Theorem}

$4a - 10 + 3a + 32.5 = 180$ \hspace{1cm} \text{Substitution}

$7a + 22.5 = 180$ \hspace{1cm} \text{Combine like terms.}

$7a = 157.5$ \hspace{1cm} \text{Subtract 22.5 from each side.}

$a = 22.5$ \hspace{1cm} \text{Divide each side by 7.}

If $a = 22.5$, then $m\angle 1 = 80$ and $m\angle 3 = 100$.

Because $\angle 1 \equiv \angle 2$ and $\angle 3 \equiv \angle 4$, $m\angle 2 = 80$ and $m\angle 4 = 100$. 

www.geometryonline.com/extra_examples

Lesson 8-6 Trapezoids 441
**Concept Check**

1. **List** the minimum requirements to show that a quadrilateral is a trapezoid.
2. **Make a chart** comparing the characteristics of the diagonals of a trapezoid, a rectangle, a square, and a rhombus. *(Hint: Use the types of quadrilaterals as column headings and the properties of diagonals as row headings.)*
3. **OPEN ENDED** Draw a trapezoid and an isosceles trapezoid. Draw the median for each. Is the median parallel to the bases in both trapezoids?

**Guided Practice**

**COORDINATE GEOMETRY**  
QRST is a quadrilateral with vertices Q(−3, 2), R(−1, 6), S(4, 6), T(6, 2).

4. Verify that QRST is a trapezoid.
5. Determine whether QRST is an isosceles trapezoid. Explain.

6. **PROOF** CDFG is an isosceles trapezoid with bases \( \overline{CD} \) and \( \overline{FG} \). Write a flow proof to prove \( \angle DGF \equiv \angle CFG \).

7. **ALGEBRA** EFGH is an isosceles trapezoid with median \( \overline{YZ} \). If \( EF = 3x + 8 \), \( HG = 4x - 10 \), and \( YZ = 13 \), find \( x \).

**Application**

8. **PHOTOGRAPHY** Photographs can show a building in a perspective that makes it appear to be a different shape. Identify the types of quadrilaterals in the photograph.

**Practice and Apply**

**COORDINATE GEOMETRY**  
For each quadrilateral whose vertices are given,  

a. verify that the quadrilateral is a trapezoid, and  
b. determine whether the figure is an isosceles trapezoid.

9. \( A(−3, 3), B(−4, −1), C(5, −1), D(2, 3) \)
10. \( G(−5, −4), H(5, 4), J(0, 5), K(−5, 1) \)
11. \( C(−1, 1), D(−5, −3), E(−4, −10), F(6, 0) \)
12. \( Q(−12, 1), R(−9, 4), S(−4, 3), T(−11, −4) \)

**ALGEBRA**  
Find the missing measure(s) for the given trapezoid.

13. For trapezoid DEGH, \( X \) and \( Y \) are midpoints of the legs. Find \( DE \).
14. For trapezoid RSTV, \( A \) and \( B \) are midpoints of the legs. Find \( VT \).
15. For isosceles trapezoid $XYZW$, find the length of the median, $m \angle W$, and $m \angle Z$.

For Exercises 17 and 18, use trapezoid $QRST$.

17. Let $\overline{GH}$ be the median of $RSBA$. Find $GH$.

18. Let $\overline{JK}$ be the median of $ABTQ$. Find $JK$.

19. **ALGEBRA** $JKLM$ is a trapezoid with $JKLM || LM$ and median $\overline{RP}$. Find $RP$ if $JK = 2(x + 3)$, $RP = 5 + x$, and $ML = \frac{1}{2}x - 1$.

20. **DESIGN** The bagua is a tool used in Feng Shui design. This bagua consists of two regular octagons centered around a yin-yang symbol. How can you determine the type of quadrilaterals in the bagua?

21. **SEWING** Madison is making a valance for a window treatment. She is using striped fabric cut on the bias, or diagonal, to create a chevron pattern. Identify the polygons formed in the fabric.

22. **COORDINATE GEOMETRY** Determine whether each figure is a trapezoid, a parallelogram, a square, a rhombus, or a quadrilateral. Choose the most specific term. Explain.

23. **COORDINATE GEOMETRY** For Exercises 26–28, refer to quadrilateral $PQRS$.

24. Determine whether the figure is a trapezoid. If so, is it isosceles? Explain.

25. Find the coordinates of the midpoints of $\overline{PQ}$ and $\overline{RS}$, and label them $A$ and $B$.

26. Find $AB$ without using the Distance Formula.
COORDINATE GEOMETRY  For Exercises 29–31, refer to quadrilateral DEFG.

29. Determine whether the figure is a trapezoid. If so, is it isosceles? Explain.
30. Find the coordinates of the midpoints of DE and EF, and label them W and V.
31. Find WV without using the Distance Formula.

PROOF  Write a flow proof.

32. Given: \( \overline{HJ} \parallel \overline{GK}, \triangle HKG \cong \triangle JKG \)
   Prove: \( \triangle GHJ \) is an isosceles trapezoid.

33. Given: \( \triangle TXZ \cong \triangle YXZ \)
   Prove: \( XYZW \) is a trapezoid.

34. Given: \( \triangle YXZ \)
   Prove: \( \triangle PWX \) is isosceles.

35. Given: \( E \) and \( C \) are midpoints of \( AD \) and \( DB \), \( AD = DB \)
   Prove: \( ABCE \) is an isosceles trapezoid.

36. Write a paragraph proof of Theorem 8.18.

CONSTRUCTION  Use a compass and straightedge to construct each figure.

37. an isosceles trapezoid
38. trapezoid with a median 2 centimeters long

39. CRITICAL THINKING  In \( RSTV \), \( RS = 6 \), \( VT = 3 \), and \( RX \) is twice the length of \( XV \). Find \( XY \).

40. CRITICAL THINKING  Is it possible for an isosceles trapezoid to have two right angles? Explain.

41. WRITING IN MATH  Answer the question that was posed at the beginning of the lesson.

   How are trapezoids used in architecture?
   Include the following in your answer:
   • the characteristics of a trapezoid,
   • other examples of trapezoids in architecture.

42. SHORT RESPONSE  What type of quadrilateral is \( WXYZ \)? Justify your answer.
43. **ALGEBRA** In the figure, which point lies within the shaded region?

- A  (−2, 4)
- B  (−1, 3)
- C  (1, −3)
- D  (2, −4)

![Graph with shaded region and points]

**Maintain Your Skills**

**Mixed Review**

**ALGEBRA** In rhombus $LMPQ$, $\angle QLM = 2x^2 - 10$, $\angle QPM = 8x$, and $MP = 10$. *(Lesson 8-5)*

44. Find $\angle LPQ$.  
45. Find $QL$.  
46. Find $\angle LQP$.  
47. Find $\angle LQM$.

**COORDINATE GEOMETRY** For Exercises 49–51, refer to quadrilateral $RSTV$. *(Lesson 8-4)*

49. Find $RS$ and $TV$.  
50. Find the coordinates of the midpoints of $\overline{RT}$ and $\overline{SV}$.  
51. Is $RSTV$ a rectangle? Explain.

**Solve each proportion.** *(Lesson 6-1)*

52. $\frac{16}{38} = \frac{24}{y}$
53. $\frac{y}{6} = \frac{17}{30}$
54. $\frac{5}{y + 4} = \frac{20}{28}$
55. $\frac{2y}{9} = \frac{52}{36}$

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Write an expression for the slope of the segment given the coordinates of the endpoints. *(To review slope, see Lesson 3-3.)*

56. $(0, a), (−a, 2a)$
57. $(−a, b), (a, b)$
58. $(c, c), (c, d)$
59. $(a, −b), (2a, b)$
60. $(3a, 2b), (b, −a)$
61. $(b, c), (−b, −c)$

**Practice Quiz 2**

Quadrilateral $ABCD$ is a rectangle. *(Lesson 8-4)*

1. Find $x$.
2. Find $y$.

3. **COORDINATE GEOMETRY** Determine whether $MNPQ$ is a rhombus, a rectangle, or a square for $M(−5, −3), N(−2, 3), P(−2, −9)$, and $Q(1, −3)$. List all that apply. Explain. *(Lesson 8-5)*

For trapezoid $TRSV$, $M$ and $N$ are midpoints of the legs. *(Lesson 8-6)*

4. If $VS = 21$ and $TR = 44$, find $MN$.
5. If $TR = 32$ and $MN = 25$, find $VS$.  

![Trapezoid TRSV with midpoints M and N]
Hierarchy of Polygons

A hierarchy is a ranking of classes or sets of things. Examples of some classes of polygons are rectangles, rhombi, trapezoids, parallelograms, squares, and quadrilaterals. These classes are arranged in the hierarchy below.

```
Polygons
   
Quadrilaterals
   
Parallelograms  Kites  Trapezoids
   
Rectangles  Rhombi
   
Squares
```

Use the following information to help read the hierarchy diagram.

- The class that is the broadest is listed first, followed by the other classes in order. For example, polygons is the broadest class in the hierarchy diagram above, and squares is a very specific class.

- Each class is contained within any class linked above it in the hierarchy. For example, all squares are also rhombi, rectangles, parallelograms, quadrilaterals, and polygons. However, an isosceles trapezoid is not a square or a kite.

- Some, but not all, elements of each class are contained within lower classes in the hierarchy. For example, some trapezoids are isosceles trapezoids, and some rectangles are squares.

Reading to Learn

Refer to the hierarchy diagram at the right. Write true, false, or not enough information for each statement.

1. All mogs are jums.
2. Some jibs are jums.
3. All lems are jums.
4. Some wibs are jums.
5. All mogs are bips.
6. Draw a hierarchy diagram to show these classes: equilateral triangles, polygons, isosceles triangles, triangles, and scalene triangles.
8-7
Coordinate Proof With Quadrilaterals

**What You’ll Learn**
- Position and label quadrilaterals for use in coordinate proofs.
- Prove theorems using coordinate proofs.

**How can you use a coordinate plane to prove theorems about quadrilaterals?**

In Chapter 4, you learned that variable coordinates can be assigned to the vertices of triangles. Then the Distance and Midpoint Formulas and coordinate proofs were used to prove theorems. The same is true for quadrilaterals.

**POSITION FIGURES** The first step to using a coordinate proof is to place the figure on the coordinate plane. The placement of the figure can simplify the steps of the proof.

**Example 1 Positioning a Square**

Position and label a square with sides \( a \) units long on the coordinate plane.
- Let \( A, B, C, \) and \( D \) be vertices of a square with sides \( a \) units long.
- Place the square with vertex \( A \) at the origin, \( \overline{AB} \) along the positive \( x \)-axis, and \( \overline{AD} \) along the \( y \)-axis. Label the vertices \( A, B, C, \) and \( D \).
- The \( y \)-coordinate of \( B \) is 0 because the vertex is on the \( x \)-axis. Since the side length is \( a \), the \( x \)-coordinate is \( a \).
- \( D \) is on the \( y \)-axis so the \( x \)-coordinate is 0. The \( y \)-coordinate is \( 0 + a \) or \( a \).
- The \( x \)-coordinate of \( C \) is also \( a \). The \( y \)-coordinate is \( 0 + a \) or \( a \) because the side \( \overline{BC} \) is \( a \) units long.

Some examples of quadrilaterals placed on the coordinate plane are given below. Notice how the figures have been placed so the coordinates of the vertices are as simple as possible.
PROVE THEOREMS  Once a figure has been placed on the coordinate plane, we can prove theorems using the Slope, Midpoint, and Distance Formulas.

Example 2  Find Missing Coordinates

Name the missing coordinates for the parallelogram. Opposite sides of a parallelogram are congruent and parallel. So, the \( y \)-coordinate of \( D \) is \( a \).

The length of \( AB \) is \( b \), and the length of \( DC \) is \( b \). So, the \( x \)-coordinate of \( D \) is \( (b + c) - b \) or \( c \).

The coordinates of \( D \) are \( (c, a) \).

Geometry Software Investigation

Quadrilaterals

Model

- Use The Geometer’s Sketchpad to draw a quadrilateral \( ABCD \) with no two sides parallel or congruent.
- Construct the midpoints of each side.
- Draw the quadrilateral formed by the midpoints of the segments.

Analyze

1. Measure each side of the quadrilateral determined by the midpoints of \( ABCD \).
2. What type of quadrilateral is formed by the midpoints? Justify your answer.

In this activity, you discover that the quadrilateral formed from the midpoints of any quadrilateral is a parallelogram. You will prove this in Exercise 22.

Example 3  Coordinate Proof

Place a square on a coordinate plane. Label the midpoints of the sides, \( M, N, P, \) and \( Q \). Write a coordinate proof to prove that \( MNPQ \) is a square.

The first step is to position a square on the coordinate plane. Label the vertices to make computations as simple as possible.

Given: \( ABCD \) is a square. \( M, N, P, \) and \( Q \) are midpoints.

Prove: \( MNPQ \) is a square.

Proof:

By the Midpoint Formula, the coordinates of \( M, N, P, \) and \( Q \) are as follows.

\[
M \left( \frac{2a + 0}{2}, \frac{0 + 0}{2} \right) = (a, 0) \quad N \left( \frac{2a + 2a}{2}, \frac{2a + 0}{2} \right) = (2a, a) \\
P \left( \frac{0 + 2a}{2}, \frac{2a + 2a}{2} \right) = (a, 2a) \quad Q \left( \frac{0 + 0}{2}, \frac{0 + 2a}{2} \right) = (0, a)
\]
Find the slopes of $\overline{QP}, \overline{MN}, \overline{QM},$ and $\overline{PN}$.

\[
\text{slope of } \overline{QP} = \frac{2a - a}{a - 0} \text{ or } 1 \\
\text{slope of } \overline{MN} = \frac{a - 0}{2a - a} \text{ or } 1 \\
\text{slope of } \overline{QM} = \frac{a - 0}{0 - a} \text{ or } -1 \\
\text{slope of } \overline{PN} = \frac{2a - a}{a - 2a} \text{ or } -1
\]

Each pair of opposite sides is parallel, so they have the same slope. Consecutive sides form right angles because their slopes are negative reciprocals.

Use the Distance Formula to find the length of $\overline{QP}$ and $\overline{QM}$.

\[
\overline{QP} = \sqrt{(a - 0)^2 + (2a - a)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} \text{ or } a\sqrt{2} \\
\overline{QM} = \sqrt{(a - 0)^2 + (0 - a)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} \text{ or } a\sqrt{2}
\]

$\text{MNPQ}$ is a square because each pair of opposite sides is parallel, and consecutive sides form right angles and are congruent.

**Example 4** Properties of Quadrilaterals

**PARKING** Write a coordinate proof to prove that the sides of the parking space are parallel.

**Given:** $A(0, 0), B(8, 0), C(14, 14), D(6, 14)$

**Prove:** $\overline{AD} \parallel \overline{BC}$

**Proof:**

\[
\text{slope of } \overline{AD} = \frac{14 - 0}{6 - 0} \text{ or } \frac{7}{3} \\
\text{slope of } \overline{BC} = \frac{14 - 0}{14 - 8} \text{ or } \frac{7}{3}
\]

Since $\overline{AD}$ and $\overline{BC}$ have the same slope, they are parallel.

**Check for Understanding**

**Concept Check**

1. Explain how to position a quadrilateral to simplify the steps of the proof.

2. OPEN ENDED Position and label a trapezoid with two vertices on the $y$-axis.

**Guided Practice** Position and label the quadrilateral on the coordinate plane.

3. rectangle with length $a$ units and height $a + b$ units

Name the missing coordinates for each quadrilateral.

4.

\[
\begin{align*}
D(0, a) & \quad O(0, 0) \\
C(?, ?) & \quad B(a, 0)
\end{align*}
\]

5.

\[
\begin{align*}
D(?, ?) & \quad O(0, 0) \\
C(a + c, b) & \quad B(a, 0)
\end{align*}
\]

Write a coordinate proof for each statement.

6. The diagonals of a parallelogram bisect each other.

7. The diagonals of a square are perpendicular.
8. **STATES** The state of Tennessee can be separated into two shapes that resemble quadrilaterals. Write a coordinate proof to prove that \(DEFG\) is a trapezoid. All measures are approximate and given in kilometers.

![Geometric Diagram of Tennessee State]

**Practice and Apply**

Position and label each quadrilateral on the coordinate plane.

9. isosceles trapezoid with height \(c\) units, bases \(a\) units and \(a + 2b\) units

10. parallelogram with side length \(c\) units and height \(b\) units

Name the missing coordinates for each quadrilateral.

11.  

12.  

13.  

14.  

15.  

16.  

Position and label each figure on the coordinate plane. Then write a coordinate proof for each of the following.

17. The diagonals of a rectangle are congruent.

18. If the diagonals of a parallelogram are congruent, then it is a rectangle.

19. The diagonals of an isosceles trapezoid are congruent.

20. The median of a trapezoid is parallel to the bases.

21. The segments joining the midpoints of the sides of a rectangle form a rhombus.

22. The segments joining the midpoints of the sides of a quadrilateral form a parallelogram.

23. **CRITICAL THINKING** \(A\) has coordinates \((0, 0)\), and \(B\) has coordinates \((a, b)\). Find the coordinates of \(C\) and \(D\) so \(ABCD\) is an isosceles trapezoid.
ARCHITECTURE  For Exercises 24–26, use the following information.
The Leaning Tower of Pisa is approximately 60 meters tall, from base to belfry.
The tower leans about 5.5° so the top level is 4.5 meters over the first level.
24. Position and label the tower on a coordinate plane.
25. Is it possible to write a coordinate proof to prove that the sides of the tower are parallel? Explain.
26. From the given information, what conclusion can be drawn?
27. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How is the coordinate plane used in proofs?
Include the following in your answer:
• guidelines for placing a figure on a coordinate grid, and
• an example of a theorem from this chapter that could be proved using the coordinate plane.

28. In the figure, \(ABCD\) is a parallelogram. What are the coordinates of point \(D\)?
\[
A \quad (a, c + b) \quad B \quad (c + b, a) \\
C \quad (b - c, a) \quad D \quad (c - b, a)
\]
29. **ALGEBRA** If \(p = -5\), then \(5 - p^2 - p = \_\_\_\_\_.\)
\[
A \quad -15 \quad B \quad -5 \\
C \quad 10 \quad D \quad 30
\]

**ARCHITECTURE**

The tower is also sinking. In 1838, the foundation was excavated to reveal the bases of the columns.

**Source:** www.torre.duomo.pisa.it

**Standardized Test Practice**

**Mixed Review**

30. **PROOF** Write a two-column proof. *(Lesson 8-6)*

**Given:** \(MNOP\) is a trapezoid with bases \(MN\) and \(OP\).
\(MN \cong QO\)

**Prove:** \(MNOQ\) is a parallelogram.

\(JKLM\) is a rectangle. \(MLPR\) is a rhombus. \(\angle JMK \cong \angle RMP\), \(m\angle JMK = 55^\circ\), and \(m\angle MRP = 70^\circ\). *(Lesson 8-5)*

31. Find \(m\angle MPR\).
32. Find \(m\angle KML\).
33. Find \(m\angle KLP\).

Find the geometric mean between each pair of numbers. *(Lesson 7-1)*

34. 7 and 14  
35. \(2\sqrt{5}\) and \(6\sqrt{5}\)

**Write an expression relating the given pair of angle measures.** *(Lesson 5-5)*

36. \(m\angle W VX\), \(m\angle VXY\)
37. \(m\angle X VZ\), \(m\angle VXZ\)
38. \(m\angle X Y V\), \(m\angle VXY\)
39. \(m\angle X Z Y\), \(m\angle X Z V\)

**More About...**

Architecture

The tower is also sinking. In 1838, the foundation was excavated to reveal the bases of the columns.

**Source:** www.torre.duomo.pisa.it

**Maintain Your Skills**

30. **PROOF** Write a two-column proof. *(Lesson 8-6)*

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\(MN \cong QO\)

**Prove:** \(MNOQ\) is a parallelogram.

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31. Find \(m\angle MPR\).
32. Find \(m\angle KML\).
33. Find \(m\angle KLP\).

Find the geometric mean between each pair of numbers. *(Lesson 7-1)*

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35. \(2\sqrt{5}\) and \(6\sqrt{5}\)

**Write an expression relating the given pair of angle measures.** *(Lesson 5-5)*

36. \(m\angle W VX\), \(m\angle VXY\)
37. \(m\angle X VZ\), \(m\angle VXZ\)
38. \(m\angle X Y V\), \(m\angle VXY\)
39. \(m\angle X Z Y\), \(m\angle X Z V\)
**Vocabulary and Concept Check**

- diagonal (p. 404)
- median (p. 440)
- rhombus (p. 431)
- isosceles trapezoid (p. 439)
- parallelogram (p. 411)
- square (p. 432)
- kites (p. 438)
- rectangle (p. 424)
- trapezoid (p. 439)

A complete list of postulates and theorems can be found on pages R1–R8.

**Exercises**  State whether each sentence is true or false. If false, replace the underlined term to make a true sentence.

1. The diagonals of a rhombus are perpendicular.
2. All squares are rectangles.
3. If a parallelogram is a rhombus, then the diagonals are congruent.
4. Every parallelogram is a quadrilateral.
5. A(n) rhombus is a quadrilateral with exactly one pair of parallel sides.
6. Each diagonal of a rectangle bisects a pair of opposite angles.
7. If a quadrilateral is both a rhombus and a rectangle, then it is a square.
8. Both pairs of base angles in a(n) isosceles trapezoid are congruent.

---

**Lesson-by-Lesson Review**

### 8-1 Angles of Polygons

**Concept Summary**
- If a convex polygon has \( n \) sides and the sum of the measures of its interior angles is \( S \), then \( S = 180(n - 2) \).
- The sum of the measures of the exterior angles of a convex polygon is 360.

**Example**

Find the measure of an interior angle of a regular decagon.

\[
S = 180(n - 2) \quad \text{Interior Angle Sum Theorem}
\]

\[
= 180(10 - 2) \quad n = 10
\]

\[
= 180(8) \text{ or } 1440 \quad \text{Simplify.}
\]

The measure of each interior angle is \( 1440 \div 10, \text{ or } 144 \).

**Exercises**

Find the measure of each interior angle of a regular polygon given the number of sides.  
*See Example 1 on page 405.*

9. 6 10. 15 11. 4 12. 20

**ALGEBRA**

Find the measure of each interior angle.  
*See Example 3 on page 405.*

13.  

14.
8-2

Parallelograms

Concept Summary
- In a parallelogram, opposite sides are parallel and congruent, opposite angles are congruent, and consecutive angles are supplementary.
- The diagonals of a parallelogram bisect each other.

Example
WXYZ is a parallelogram. Find $m\angle YZW$ and $m\angle XWZ$.

\[
\begin{align*}
\text{Opp. of } \square \text{ are } \cong. \\
m\angle YZW &= m\angle WXY \\
m\angle YZW &= 82 + 33 \text{ or } 115 \\
m\angle XWZ + m\angle WXY &= 180 \\
m\angle XWZ + (82 + 33) &= 180 \\
m\angle XWZ + 115 &= 180 \\
m\angle XWZ &= 65 \\
\text{Simplify.}
\end{align*}
\]

Exercises
Use $\square ABCD$ to find each measure.

See Example 2 on page 413.

15. $m\angle BCD$  
16. $AF$  
17. $m\angle BDC$  
18. $BC$  
19. $CD$  
20. $m\angle ADC$

8-3

Tests for Parallelograms

Concept Summary
A quadrilateral is a parallelogram if any one of the following is true.
- Both pairs of opposite sides are parallel and congruent.
- Both pairs of opposite angles are congruent.
- Diagonals bisect each other.
- A pair of opposite sides is both parallel and congruent.

Example
COORDINATE GEOMETRY Determine whether the figure with vertices $A(-5, 3), B(-1, 5), C(6, 1)$, and $D(2, -1)$ is a parallelogram. Use the Distance and Slope Formulas.

\[
\begin{align*}
AB &= \sqrt{(-5 - (-1))^2 + (3 - 5)^2} \\
&= \sqrt{(-4)^2 + (-2)^2} \text{ or } \sqrt{20} \\
CD &= \sqrt{(6 - 2)^2 + [1 - (-1)]^2} \\
&= \sqrt{4^2 + 2^2} \text{ or } \sqrt{20} \\
\text{Since } AB &= CD, AB \cong CD. \\
\text{slope of } AB &= \frac{5 - 3}{-1 - (-5)} \text{ or } \frac{1}{2} \\
\text{slope of } CD &= \frac{-1 - 1}{2 - 6} \text{ or } \frac{1}{2} \\
AB \text{ and } CD \text{ have the same slope, so they are parallel. Since one pair of opposite sides is congruent and parallel, } AB \text{ and } CD \text{ is a parallelogram.}
\end{align*}
\]
Exercises Determine whether the figure with the given vertices is a parallelogram. Use the method indicated. See Example 5 on page 420.
21. \( A(-2, 5), B(4, 4), C(6, -3), D(-1, -2); \) Distance Formula
22. \( H(0, 4), J(-4, 6), K(5, 6), L(9, 4); \) Midpoint Formula
23. \( S(-2, -1), T(2, 5), V(-10, 13), W(-14, 7); \) Slope Formula

Rectangles

Concept Summary
• A rectangle is a quadrilateral with four right angles and congruent diagonals.
• If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Example
Quadrilateral \( KLMN \) is a rectangle. If \( PL = x^2 - 1 \) and \( PM = 4x + 11 \), find \( x \).
The diagonals of a rectangle are congruent and bisect each other, so \( PL \equiv PM \).

\[
\begin{align*}
PL & \equiv PM & \text{Diag. are } \equiv \text{ and bisect each other.} \\
PL & = PM & \text{Def. of } \equiv \text{ angles} \\
x^2 - 1 & = 4x + 11 & \text{Substitution} \\
x^2 - 1 - 4x & = 11 & \text{Subtract } 4x \text{ from each side.} \\
x^2 - 4x - 12 & = 0 & \text{Subtract } 11 \text{ from each side.} \\
(x + 2)(x - 6) & = 0 & \text{Factor.} \\
x + 2 & = 0 & x - 6 = 0 \\
x & = -2 & x = 6
\end{align*}
\]
The value of \( x \) is \(-2\) or \(6\).

Exercises \( ABCD \) is a rectangle. See Examples 1 and 2 on pages 425 and 426.
24. If \( AC = 26 \) and \( AF = 2x + 7 \), find \( AF \).
25. If \( m \angle 1 = 52 \) and \( m \angle 2 = 16x - 12 \), find \( m \angle 2 \).
26. If \( CF = 4x + 1 \) and \( DF = x + 13 \), find \( x \).
27. If \( m \angle 2 = 70 - 4x \) and \( m \angle 5 = 18x - 8 \), find \( m \angle 5 \).

COORDINATE GEOMETRY Determine whether \( RSTV \) is a rectangle given each set of vertices. Justify your answer. See Example 4 on pages 426 and 427.
28. \( R(-3, -5), S(0, -5), T(0, 4), V(3, 4) \)
29. \( R(0, 0), S(6, 3), T(-2, 4), V(4, 7) \)
Rhombi and Squares

Concept Summary

- A rhombus is a quadrilateral with each side congruent, diagonals that are perpendicular, and each diagonal bisecting a pair of opposite angles.
- A quadrilateral that is both a rhombus and a rectangle is a square.

Use rhombus \(JKLM\) to find \(m \angle JMK\) and \(m \angle KJM\).

The opposite sides of a rhombus are parallel, so \(KL \parallel JM\). \(\angle JMK \cong \angle LKM\) because alternate interior angles are congruent.

\[
m \angle JMK = m \angle LKM \quad \text{Definition of congruence}
\]

\[
= 28 \quad \text{Substitution}
\]

The diagonals of a rhombus bisect the angles, so \(\angle JKM \cong \angle LKM\).

\[
m \angle KJM + m \angle JKL = 180 \quad \text{Cons. } \triangle \text{ in } \square \text{ are suppl.}
\]

\[
m \angle KJM + (m \angle JKM + m \angle LKM) = 180
m \angle KJM + (28 + 28) = 180 \quad \text{Substitution}
m \angle KJM + 56 = 180 \quad \text{Add.}
m \angle KJM = 124 \quad \text{Subtract 56 from each side.}
\]

Exercises

Use rhombus \(ABCD\) with \(m \angle 1 = 2x + 20, m \angle 2 = 5x - 4, AC = 15,\) and \(m \angle 3 = y^2 + 26\). See Example 2 on page 432.

30. Find \(x\).
31. Find \(AF\).
32. Find \(y\).

Trapezoids

Concept Summary

- In an isosceles trapezoid, both pairs of base angles are congruent and the diagonals are congruent.
- The median of a trapezoid is parallel to the bases, and its measure is one-half the sum of the measures of the bases.

Example

\(RSTV\) is a trapezoid with bases \(RV\) and \(ST\) and median \(MN\). Find \(x\) if \(MN = 60, ST = 4x - 1,\) and \(RV = 6x + 11\).

\[
MN = \frac{1}{2} (ST + RV)
\]

\[
60 = \frac{1}{2} [(4x - 1) + (6x + 11)] \quad \text{Substitution}
\]

\[
120 = 4x - 1 + 6x + 11 \quad \text{Multiply each side by 2.}
\]

\[
120 = 10x + 10 \quad \text{Simplify.}
\]

\[
110 = 10x \quad \text{Subtract 10 from each side.}
\]

\[
x = 11 \quad \text{Divide each side by 10.}
\]
Exercises  Find the missing value for the given trapezoid.
See Example 4 on page 441.

33. For isosceles trapezoid $ABCD$, $X$ and $Y$ are midpoints of the legs. Find $m\angle XBC$ if $m\angle ADY = 78$.  

34. For trapezoid $JKLM$, $A$ and $B$ are midpoints of the legs. If $AB = 57$ and $KL = 21$, find $JM$.

Coordinate Proof with Quadrilaterals

Concept Summary

- Position a quadrilateral so that a vertex is at the origin and at least one side lies along an axis.

Example

Position and label rhombus $RSTV$ on the coordinate plane. Then write a coordinate proof to prove that each pair of opposite sides is parallel.

First, draw rhombus $RSTV$ on the coordinate plane. Label the coordinates of the vertices.

Given: $RSTV$ is a rhombus.

Prove: $RV \parallel ST$, $RS \parallel VT$

Proof:

slope of $RV = \frac{c - 0}{b - 0}$ or $\frac{c}{b}$

slope of $ST = \frac{c - 0}{(a + b) - a}$ or $\frac{c}{b}$

slope of $RS = \frac{0 - 0}{a - 0}$ or $0$

slope of $VT = \frac{c - c}{(a + b) - b}$ or $0$

$RV$ and $ST$ have the same slope. So $RV \parallel ST$. $RS$ and $VT$ have the same slope, and $RS \parallel VT$.

Exercises  Position and label each figure on the coordinate plane. Then write a coordinate proof for each of the following.  See Example 3 on pages 448 and 449.

35. The diagonals of a square are perpendicular.

36. A diagonal separates a parallelogram into two congruent triangles.

Name the missing coordinates for each quadrilateral.  See Example 2 on page 448.

37.

38.
Determine whether each conditional is true or false. If false, draw a counterexample.
1. If a quadrilateral has four right angles, then it is a rectangle.
2. If a quadrilateral has all four sides congruent, then it is a square.
3. If the diagonals of a quadrilateral are perpendicular, then it is a rhombus.

Complete each statement about \( \square FGHK \). Justify your answer.
4. \( HK \cong ? \).
5. \( \angle FKH \cong ? \).
6. \( \angle FKH \equiv ? \).
7. \( GH \parallel ? \).

Determine whether the figure with the given vertices is a parallelogram. Justify your answer.
8. \( A(4, 3), B(6, 0), C(4, -8), D(2, -5) \)
9. \( A(-2, 6), B(11, 2), V(3, 8), W(-1, 3) \)
10. \( F(7, -3), G(4, -2), H(6, 4), J(12, 2) \)
11. \( W(-4, 2), X(-3, 6), Y(2, 7), Z(1, 3) \)

ALGEBRA \( QRST \) is a rectangle.
12. If \( QP = 3x + 11 \) and \( PS = 4x + 8 \), find \( QS \).
13. If \( m\angle QTR = 2x^2 - 7 \) and \( m\angle SRT = x^2 + 18 \), find \( m\angle QTR \).

COORDINATE GEOMETRY Determine whether \( \square ABCD \) is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.
14. \( A(12, 0), B(6, -6), C(0, 0), D(6, 6) \)
15. \( A(-2, 4), B(5, 6), C(12, 4), D(5, 2) \)

Name the missing coordinates for each quadrilateral.
16. \( N(0, c), P(?, ?) \)
17. \( A(0, 0), B(a + 2b, 0), D(?, ?), C(?, ?) \)

Position and label a trapezoid on the coordinate plane. Write a coordinate proof to prove that the median is parallel to each base.

SAILING Many large sailboats have a keel to keep the boat stable in high winds. A keel is shaped like a trapezoid with its top and bottom parallel. If the root chord is 9.8 feet and the tip chord is 7.4 feet, find the length of the mid-chord.

STANDARDIZED TEST PRACTICE The measure of an interior angle of a regular polygon is 108. Find the number of sides.

\[ \text{A} \quad 8 \quad \text{B} \quad 6 \quad \text{C} \quad 5 \quad \text{D} \quad 3 \]
1. A trucking company wants to purchase a ramp to use when loading heavy objects onto a truck. The closest that the truck can get to the loading area is 5 meters. The height from the ground to the bed of the truck is 3 meters. To the nearest meter, what should the length of the ramp be? (Lesson 1-3)

- [ ] 4 m
- [ ] 5 m
- [ ] 6 m
- [ ] 7 m

2. Which of the following is the contrapositive of the statement below? (Lesson 2-3)

If an astronaut is in orbit, then he or she is weightless.

- [A] If an astronaut is weightless, then he or she is in orbit.
- [B] If an astronaut is not in orbit, then he or she is not weightless.
- [C] If an astronaut is on Earth, then he or she is weightless.
- [D] If an astronaut is not weightless, then he or she is not in orbit.

3. Rectangle QRST measures 7 centimeters long and 4 centimeters wide. Which of the following could be the dimensions of a rectangle similar to rectangle QRST? (Lesson 6-2)

- [A] 28 cm by 14 cm
- [B] 21 cm by 12 cm
- [C] 14 cm by 4 cm
- [D] 7 cm by 8 cm

4. A 24 foot ladder, leaning against a house, forms a 60° angle with the ground. How far up the side of the house does the ladder reach? (Lesson 7-3)

- [A] 12 ft
- [B] $12\sqrt{2}$ ft
- [C] $12\sqrt{3}$ ft
- [D] 20 ft

5. In rectangle JKLM shown below, JL and MK are diagonals. If $JL = 2x + 5$ and $KM = 4x - 11$, what is $x$? (Lesson 8-4)

- [A] 10
- [B] 8
- [C] 6
- [D] 5

6. Joaquin bought a set of stencils for his younger sister. One of the stencils is a quadrilateral with perpendicular diagonals that bisect each other, but are not congruent. What kind of quadrilateral is this piece? (Lesson 8-5)

- [A] square
- [B] rectangle
- [C] rhombus
- [D] trapezoid

7. In the diagram below, ABCD is a trapezoid with diagonals AC and BD intersecting at point E.

Which statement is true? (Lesson 8-6)

- [A] $\overline{AB}$ is parallel to $\overline{CD}$.
- [B] $\angle ADC$ is congruent to $\angle BCD$.
- [C] $\overline{CE}$ is congruent to $\overline{DE}$.
- [D] $\overline{AC}$ and $\overline{BD}$ bisect each other.
Part 2  Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. At what point does the graph of \( y = -4x + 5 \) cross the x-axis on a coordinate plane? (Prerequisite Skill)

9. Candace and Julio are planning to see a movie together. They decide to meet at the house that is closer to the theater. From the locations shown on the diagram, whose house is closer to the theater? (Lesson 5-3)

10. In the diagram, \( \overline{CE} \) is the mast of a sailboat with sail \( \triangle ABC \).

Marcia wants to calculate the length, in feet, of the mast. Write an equation in which the geometric mean is represented by \( x \). (Lesson 7-1)

Part 3  Open Ended

Record your answers on a sheet of paper. Show your work.

12. On the tenth hole of a golf course, a sand trap is located right before the green at point \( M \). Matt is standing 126 yards away from the green at point \( N \). Quintashia is standing 120 yards away from the beginning of the sand trap at point \( Q \).

a. Explain why \( \triangle MNR \) is similar to \( \triangle PQR \). (Lesson 6-3)

b. Write and solve a proportion to find the distance across the sand trap, \( a \). (Lesson 6-3)

13. Quadrilateral \( ABCD \) has vertices with coordinates: \( A(0, 0) \), \( B(a, 0) \), \( C(a + b, c) \), and \( D(b, c) \).

a. Position and label \( ABCD \) on the coordinate plane. Prove that \( ABCD \) is a parallelogram. (Lesson 8-2 and 8-7)

b. If \( a^2 = b^2 + c^2 \), what can you determine about the slopes of the diagonals \( \overline{AC} \) and \( \overline{BD} \)? (Lesson 8-7)

c. What kind of parallelogram is \( ABCD \)? (Lesson 8-7)